Characteristics of Multilayer Slab Waveguide Structures Comprising Metamaterials at Terahertz Frequency

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Characteristics of Multilayer Slab Waveguide
Structures Comprising Metamaterials at Terahertz Frequency

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By
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DECLARATION

I understand the nature of plagiarism, and I am aware of the University’s policy on this. The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted by others elsewhere for any other degree or qualification.
DEDICATION

To the soul of my father…

To my loving and caring mother…

To my brothers and sisters…

To my devoted wife

"Om Hamza"

To my children

"Hamza–Jory–Kareem–Ameer"

I dedicate this thesis.
ACKNOWLEDGEMENT

I wish to express my profound gratitude to both my supervisors Prof. Hassan S. Ashour and Dr. Mazen M. Al-Abadla for their encouragement and guidance offered me throughout this thesis. I would like to express my deep gratitude to both of them for their constant support, help guidance, stimulating suggestions for arranging and writing this thesis and for the inspiration I experienced with him. Also, I sincerely thank all members of my family and my colleagues for their continuous support and encouragement.

Researcher
Hassan M.A. Sultana
ABSTRACT

Dispersion properties of guided modes in asymmetric three-layers and four-layers slab waveguide structures are studied. The dispersion relation for the waveguide structure for both transverse electric TE and transverse magnetic TM modes in THz frequencies is investigated using transfer matrix approach. Electric field E, power flow P, confinement, and penetration depth Δx are derived and plotted.

For a three-layer case, we assume three structures: the first comprises a left-handed material LHM core, the second comprises a ENG substrate and the third has a ENG cladding. The results show that the proposed waveguide structures exhibit very high power confinement factor with low to moderate loss. Thus, the structures have a good guidance properties and are good for long distance EM wave guiding.

For a four-layer case, we assume two structures: the first having LHM guiding film, optical properties are studied for different modes order. The results show that the proposed waveguide structures exhibit low effective refractive index, and low attenuation for higher modes, thus lower order modes are suitable for long distance wave guiding.

The second structure has two LHM layers, the study concentrates on the effect of thickness on optical properties of the structures. The results show that the proposed waveguide structures exhibit low effective refractive index, very low attenuation, and high confinement factor. Thus, we have a good structure for wave guiding.

Besides that, we studied propagation characteristic of TM modes in three layers and four asymmetric structures. We found that the attenuation of TM modes is higher than TE modes.
الملخص

لقد تم في هذه الأطروحة دراسة الخصائص الضوئية للموجات المنعوجة متعددة الطبقات التي تحتوي على مواد يسارية في طبقة أو أكثر وذلك عند ترددات الثيرة حيرت ف. ولهذا في هذه الدراسة اعتماد نوعين من الموجات المنعوجة: الأولى الموجات المنعوجة عموديا على مستوى السقوط (أو الموجات المستعرضة الكهربائية) والثاني الموجات المنعوجة في مستوى السقوط (أو الموجات المنعوجة المغناطيسية). كما أجريت مقارنة بين أداء التراكيب المختلفة في الحالتين.

وقد اعتمدت هنا مصفوفة الاندماج للحصول على معادلة التشتت لتركيب ثلاثية ورباعية تحتوي على المواد اليسارية ثم طورت البرامج الوراثية لرسم محاكاة التشتت في الحالات المختلفة والمتغيرات المختلفة. وقد رسمت العلاقة لمعامل الانكسار الفعال في حالتين الحقيقية والتخيلية. لقد تم أيضاً إيجاد القدر المتغير خلال طبقات المرشدة في التوليفات المختلفة ومن ثم حساب نسبة الطاقة المتدفقة في كل طبقة وتمت دراستها مع التردد عند عدة قيم لسمك المرشدات. ولمعرفة التوهين الحاصل للموجات الكهرومغناطيسية المت MIS. في الطبقات فقد تم دراسة عمق الأذواق واعتماد على التردد خاصه. يشار أيضاً أن شكل المجال الكهربائي قد تم رسمه في التوليفات المختلفة للإطلاع على مواده بذلك لنتائج العديدة.

وقد خلصت الدراسة أن الامتصاص والتوهين داخل المرشدات المفترضة قليل بشكل عام وأن الضوء فيها محصور بصورة عالية مما يجعلها مناسبة لنقل الإشارة لمسافة طويلة. وان كانت بعض النتائج قد أظهرت امتصاصاً عالياً نسبياً.
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<tr>
<td>EM</td>
<td>Electromagnetic Metamaterials</td>
</tr>
<tr>
<td>LHM</td>
<td>Left-Handed Material</td>
</tr>
<tr>
<td>RHIM</td>
<td>Right-Handed Material</td>
</tr>
<tr>
<td>NIM</td>
<td>Negative-Index Material</td>
</tr>
<tr>
<td>DNG</td>
<td>Double Negative</td>
</tr>
<tr>
<td>DPS</td>
<td>Double Positive</td>
</tr>
<tr>
<td>TE</td>
<td>Transverse Electric</td>
</tr>
<tr>
<td>TM</td>
<td>Transverse Magnetic</td>
</tr>
<tr>
<td>THz</td>
<td>Terahertz</td>
</tr>
<tr>
<td>SRR</td>
<td>Split Ring Resonators</td>
</tr>
<tr>
<td>ENG</td>
<td>Epsilon Negative</td>
</tr>
<tr>
<td>MNG</td>
<td>Mu Negative</td>
</tr>
<tr>
<td>SNG</td>
<td>Single Negative Metamaterials</td>
</tr>
<tr>
<td>EBG</td>
<td>Electromagnetic Band Gap</td>
</tr>
<tr>
<td>ZIM</td>
<td>Zero Index of Refraction</td>
</tr>
<tr>
<td>FSS</td>
<td>Frequency Selective Surface</td>
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# LIST OF SYMBOLS

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<tr>
<td>( \varepsilon )</td>
<td>Electric permittivity of the medium</td>
</tr>
<tr>
<td>( \mu )</td>
<td>Magnetic permeability of the medium</td>
</tr>
<tr>
<td>( k )</td>
<td>Wave number</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>Damping coefficient</td>
</tr>
<tr>
<td>( F )</td>
<td>Fractional area of the unit cell</td>
</tr>
<tr>
<td>( \omega_p )</td>
<td>The plasma frequency</td>
</tr>
<tr>
<td>( \omega )</td>
<td>Angular frequency</td>
</tr>
<tr>
<td>( \omega_0 )</td>
<td>Resonance frequency</td>
</tr>
<tr>
<td>( N )</td>
<td>Effective index of the guided mode</td>
</tr>
<tr>
<td>( P )</td>
<td>Total power</td>
</tr>
<tr>
<td>( E )</td>
<td>Electric field</td>
</tr>
<tr>
<td>( H )</td>
<td>Magnetic field</td>
</tr>
<tr>
<td>( \Delta x )</td>
<td>Penetration depth</td>
</tr>
<tr>
<td>( h_{\text{eff}} )</td>
<td>Effective guide thickness</td>
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Dispersion curves: Imaginary refractive index against frequency for different thicknesses: a) $w = 20 \text{ nm}$ (solid line), b) $w = 40 \text{ nm}$ (dashed) and c) $w = 60 \text{ nm}$ (dotted), for $TE_0$ mode, $\varepsilon_0 = 2.99, \mu_1 = -1 - 0.004i$, $\varepsilon_2 = 3.38$, $\omega_p = 1.2 \times 10^{16}, \omega_0 = 1.2 \times 10^{14}, F = 0.56$.

Modal field distribution of the structure under consideration for $TE_0$, Thickness $w = 200 \text{nm}$, $\varepsilon_0 = 2.99, \mu_1 = -1 - 0.004i$, $\varepsilon_2 = 3.38$, $\omega_p = 1.2 \times 10^{16}, \omega_0 = 1.2 \times 10^{14}, F = 0.56$.

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Modal field distribution of the structure under consideration for: a) $TE_0$ (solid line), b) $TE_1$ (dashed) and c) $TE_2$ (dotted), Thickness $w = 200 \text{nm}$, $\varepsilon_0 = 2.99, \mu_1 = -1 - 0.004i$, $\varepsilon_2 = 3.38$, $\omega_p = 1.2 \times 10^{16}, \omega_0 = 1.2 \times 10^{14}, F = 0.56$.

Power flow in a three layers waveguide structures comprising LHM core.

Penetration depth ($\Delta x$) for different thicknesses: 1) $w = 100 \text{ nm}$ (solid line), 2) $w = 300 \text{ nm}$ (dashed) and 3) $w = 500 \text{ nm}$ (dotted), for $TE_0$ mode, $\varepsilon_0 = 2.99, \mu_2 = -1 - 0.004i$, $\varepsilon_1 = 3.38$, $\omega_p = 1.2 \times 10^{16}, \omega_0 = 1.2 \times 10^{14}, F = 0.56$.

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Power curves for different thicknesses: 1) $w = 100 \text{ nm}$ (solid line), 2) $w = 300 \text{ nm}$ (dashed) and 3) $w = 500 \text{ nm}$ (dotted), for $TE_0$ mode, $\varepsilon_0 = 2.99, \mu_2 = -1 - 0.004i$, $\varepsilon_1 = 3.38$, $\omega_p = 1.2 \times 10^{16}, \omega_0 = 1.2 \times 10^{14}, F = 0.56$. 

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$1.2 \times 10^{16}, \omega_0 = 1.2 \times 10^{14}, F = 0.56.$

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Fig(4.11) Dispersion curves: Im(N) against frequency for: a)h1=500nm (solid line), b) h1=550nm (dashed) and c) h1=600nm (dotted), Thickness h2=100nm, \varepsilon_0=3.21+0.02I, \varepsilon_1=-6.22+0.04I, \mu_1=-1+0.002I, \varepsilon_2=-11.18+0.06I, \mu_2=-1+0.002I, \varepsilon_3=4.14+0.03I.

Fig(4.12) Modal field distribution of the structure under last consideration for TE\textsubscript{0}.

Fig(4.13) Total power flow against frequency for: a) h1=500nm (solid line), b) h1=550nm (dashed) and c) h1=600nm (dotted), Thickness h2=100nm, \varepsilon_0=3.21+0.02I, \varepsilon_1=-6.22+0.04I, \mu_1=-
1+0.002I, \( \varepsilon_2 = -11.18 + 0.06I, \mu_2 = -1 + 0.002I, \varepsilon_3 = 4.14 + 0.03I. \)

**Fig(4.14)**  Power flow in cladding and substrate against frequency for: a) \( h_1 = 500\text{nm} \) (solid line), b) \( h_1 = 550\text{nm} \) (dashed) and c) \( h_1 = 600\text{nm} \) (dotted). Thickness \( h_2 = 100\text{nm}, \varepsilon_0 = 3.21 + 0.02I, \varepsilon_1 = -6.22 + 0.04I, \mu_1 = -1 + 0.002I, \varepsilon_2 = -11.18 + 0.06I, \mu_2 = -1 + 0.002I, \varepsilon_3 = 4.14 + 0.03I. \)

**Fig(4.15)**  Penetration depth (\( \Delta x \)) for different thicknesses: a) \( h_1 = 500\text{nm} \) (solid line), b) \( h_1 = 550\text{nm} \) (dashed) and c) \( h_1 = 600\text{nm} \) (dotted). Thickness \( h_2 = 100\text{nm}, \varepsilon_0 = 3.21 + 0.02I, \varepsilon_1 = -6.22 + 0.04I, \mu_1 = -1 + 0.002I, \varepsilon_2 = -11.18 + 0.06I, \mu_2 = -1 + 0.002I, \varepsilon_3 = 4.14 + 0.03I. \)

**Fig(5.1)**  Schematic diagram of a three-layers slab waveguide structure including LHM cladding.

**Fig(5.2)**  Dispersion curves: Real refractive index against frequency for different of thicknesses: a) \( TM_1 \) (solid line), b) \( TE_1 \) (dashed), \( w = [900 - 950]\text{nm}, \varepsilon_1 = 2.99, \mu_0 = -1 - 0.4I, \varepsilon_2 = 3.38, \omega_p = 1.2 \times 10^{16}, \omega_0 = 1.2 \times 10^{14}, F = 0.56. \)

**Fig(5.3)**  Dispersion curves: Imaginary refractive index against frequency thicknesses: a) \( TM_1 \) (solid line), b) \( TE_1 \) (dashed), \( w = [900 - 950]\text{nm}, \varepsilon_1 = 2.99, \mu_0 = -1 - 0.4I, \varepsilon_2 = 3.38, \omega_p = 1.2 \times 10^{16}, \omega_0 = 1.2 \times 10^{14}, F = 0.56. \)

**Fig(5.4)**  Modal field distribution of the structure under consideration for \( TM_1 \), Thickness \( w = 850\text{nm}, \varepsilon_1 = 2.99, \mu_0 = -1 - 0.4I, \varepsilon_2 = 3.38, \omega_p = 1.2 \times 10^{16}, \omega_0 = 1.2 \times 10^{14}, F = 0.56. \)

**Fig(5.5)**  Power curves for different thicknesses: 1) \( w = 800\text{ nm} \) (solid line), 2) \( w = 850\text{ nm} \) (dashed) and 3) \( w = 900\text{ nm} \) (dotted), for \( TM_1 \) mode, \( \varepsilon_1 = 2.99, \mu_0 = -1 - 0.4I, \varepsilon_2 = 3.38, \omega_p = 1.2 \times 10^{16}, \omega_0 = 1.2 \times 10^{14}, F = 0.56. \)

**Fig(5.6)**  Schematic diagram of Four-layers with a left-handed metamaterial.

**Fig(5.7)**  Dispersion curves: Re(N) against frequency for: a) \( TM_2 \) (solid line), b) \( TE_2 \) (dashed), Thickness \( h_1 = 800\text{nm}, h_2 = 400\text{nm}. \)

**Fig(5.8)**  Dispersion curves: Im(N) against frequency for: a) \( TM_2 \)
(solid line), b) TE₂ (dashed), Thickness \( h₁=800\text{nm} \), \( h₂=400\text{nm} \).

Fig(5.9) Modal field distribution of the structure under last consideration for TM₂.

Fig(5.10) Power flow: a) TM₁ (solid line), b) TM₂ (dashed) and c) TM₃ (dotted), Thickness \( h₁=800\text{nm} \), \( h₂=400\text{nm} \).
CHAPTER 1

Introduction to Metamaterials
Metamaterials are artificial periodic structures with lattice constants that are much smaller than the wavelength of the incident radiation. The prefix “Meta” is taken from Greek whose meaning is “beyond” or “after” [1]. “Metamaterials” have exotic properties beyond the natural occurring materials. These are the materials that extract their properties from their structure rather than the material of which they are composed of [2].

1.1 A brief Historical Review

The history of these materials began with the paper of a Russian physicist, V. G. Veselago in 1968, which predicted, theoretically, several special electromagnetic phenomena of materials with simultaneously negative permittivity and permeability [3]. Pendry et. al in 1996 first theoretically suggested and later experimentally demonstrated that a composite medium of periodically placed thin metallic wires can behave as an effective plasma medium for radiation with wavelength much larger than the spatial periodicity of the structure. For frequencies lower than a particular (plasma) frequency, the thin wire structure therefore exhibits a negative permittivity $\varepsilon$ [1, 4].

*Fig(1.1):* First experimental monodimensionally LHM structures consisted of thin continuous wires by Pandey, et al.
In 2000, Smith proved a new LHM that shows simultaneously negative permittivity and permeability and carried out microwave experiments to test its novelty. At their early experiments, they used metamaterials with repeated the unit cells of split ring resonators (SRR) and copper strips. Then, many researchers have worked on metamaterials to extract their potential in various fields [5].

1.2 Theoretical Prediction of Left-Handed Materials

To investigate the electromagnetic properties of such a medium, we first of all, we study how the electromagnetic wave behaves when \( \varepsilon < 0 \) and \( \mu < 0 \). The source-free Maxwell equations and the constitutive relations in an isotropic medium are [6]

\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} 
\]

with:

\[
\vec{B} = \mu \vec{H} \\
\vec{D} = \varepsilon \vec{E}
\]

where \( \vec{E} \) and \( \vec{H} \) are the electric and magnetic fields, respectively, \( \vec{D} \) and \( \vec{B} \) are the electric and magnetic flux densities.

The trial plane wave solutions of the electromagnetic fields are expressed in the order of \( \exp[i(\mathbf{kz} - \omega t)] \). Substituting this in (1.1) then:

\[
\mathbf{k} \times \vec{E} = \omega \mu \vec{H} \\
\mathbf{k} \times \vec{H} = -\omega \varepsilon \vec{E}
\]

It can be seen from Eq. (1.3) that when \( \varepsilon < 0 \) and \( \mu < 0 \), \( \vec{E}, \vec{H} \) and \( \mathbf{k} \) form a left-handed triplet of vectors (i.e. \( \vec{E} \times \vec{H} \)) is negative. This is the reason that the medium is named as the left-handed medium. The direction of the energy flow is given by the Poynting vector,
The non-zero Poynting vector always forms a right-handed coordinate system with $\vec{E}$ and $\vec{H}$ independent on the signs of $\varepsilon$ and $\mu$. Therefore, in a left-handed material, the wave vector, $k$, is in the opposite direction of the energy density flow, $\vec{S}$. Such a wave is called backward wave. In contrast, in a normal right-handed material (RHM), the wave vector, $k$, and the energy flow, $\vec{S}$, are in the same direction and the wave is a forward one. The directions of $E$, $H$ and $k$ for both RHM and LHM are shown in Fig.(1.2) [7].

Pendry et al [7-8] introduced permittivity $\varepsilon(\omega)$ and permeability $\mu(\omega)$ in a general form as:

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i \gamma\omega}$$

and

$$\mu(\omega) = 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 + i \gamma\omega}$$
Where $\omega_p$ is the plasma frequency, $\gamma$ is the electron scattering, $\omega$ is the operating frequency, $\omega_0$ is the resonance frequency, $\gamma$ is the electron scattering and $F$ is the fractional area of unit cell occupied by the split ring.

The behavior of Electromagnetic field is determined by the properties of the materials include. These properties define the macroscopic parameters permittivity $\varepsilon$ and permeability $\mu$ of the materials. On the basis of permittivity $\varepsilon$ and permeability $\mu$, the metamaterials are classified into the following four groups as shown in Fig.(1.3).

![Fig.(1.3): Classification of materials](image)

The materials in the first quadrant have both permittivity and permeability greater than zero ($\varepsilon > 0, \mu > 0$) are called double positive (DPS) materials. Most occurring media (e.g. dielectrics) exist under this class.

If a material has permittivity less than zero and permeability greater than zero ($\varepsilon < 0, \mu > 0$) then it is called epsilon negative (ENG) material. In certain frequency regimes, many plasmas exhibit these characteristics [8]. This class of materials is used in metamaterial antenna to increase the power efficiency and bandwidth.

Materials existing in the third quadrant have both permittivity and permeability less than zero ($\varepsilon < 0, \mu < 0$). This class of material does not exist in nature, it is
manmade material or metamaterial [1]. Such materials are called left handed materials, \( (LHM) \), or double negative material, \( (DNG) \). LHM’s follow the left handed rule because propagation of wave takes place in backward direction in the medium. Due to negative \( \mu \) and negative \( \varepsilon \) the refractive index of the medium is calculated to be negative.

If a material has permittivity greater than zero and permeability less than zero \( (\varepsilon > 0, \mu < 0) \) it is called as mu negative \( (MNG) \) material. In certain frequency regimes, some gyro tropic material exhibits these characteristics such as magnetic plasma. This type of materials is represented by the fourth quadrant in the figure.

No propagating waves can be supported in materials represented by the second and fourth quadrants, where one of the two parameters is negative and the index of refraction becomes purely imaginary. In the domain of optics, all conventional materials are confined to an extremely narrow zone around a horizontal line at \( (\mu = 1) \) in the space, as represented by the dashed line in Fig.(1.3).

1.3 Types of Metamaterials

A. Electromagnetic Metamaterials

Electromagnetic metamaterials \( (EM) \) are the materials which have a new sub section within electromagnetism and physics. \( EM \) is used for optical and microwave applications like, band-pass filters, perfect lenses, microwave couplers, beam steerers, and antenna. A metamaterial affects lesser on electromagnetic waves as compared to wavelength of electromagnetic radiation.

1. Single Negative Metamaterials

Single negative metamaterials \( (SNG) \) have either negative permittivity or negative permeability. The combination of two \( SNG \) layers into one creates another form of \( DNG \) metamaterials [9, 10]. In wave reflection experiments, slab of \( MNG \) materials and \( ENG \) materials are usually joined. Like \( DNG \) metamaterials, \( SNGs \) change their parameters like refraction index \( n \), permittivity \( \varepsilon \) and permeability \( \mu \), with change in frequency due to their dispersive nature.
2. Double Negative Metamaterials

Double negative metamaterials (DNG) are the metamaterials that have both negative permittivity and negative permeability with negative index of refraction. These are also known as negative index metamaterials (NIM), or left handed material (LHM) [11]. Other names for (DNGs) are left handed media, media with a negative refractive index, or “backward-wave media [9, 11, 12].

3. Electromagnetic Band Gap (EBG) Metamaterials

Electromagnetic band gap metamaterials control the propagation of light. They are achieved either by photonic crystals (PC), or left-handed materials (LHM). Both classes have artificial structure that control and manipulate the propagation of electromagnetic waves.

4. Bi-isotropic and Bi-anisotropic Metamaterials

Based on the independent electric and magnetic responses described by the parameters permittivity and magnetic permeability, the metamaterials are categorized into single or double negative. However in many examples of electromagnetic metamaterials, the electric field causes magnetic polarization, and the magnetic field induces an electrical polarization, i.e., magneto electric coupling. Such media are denoted as bi-isotropic media because they exhibit magneto-electric coupling which is anisotropic. They are also called bi-anisotropic [9, 13].

B. Chiral Metamaterials

Chiral metamaterials, materials exhibit optical activity, are constructed from chiral materials in which the effective parameter \( \mathbf{k} \) is non-zero. Chiral metamaterials consist of arrays of dielectric gammadions (Greek letter gamma like structures) or planar metallic on a substrate. When a linearly polarized light is incident on the array, it becomes elliptically polarized upon interaction with the gammadions with the same handedness as the gammadion itself [14, 15].

C. Terahertz Metamaterials

Terahertz metamaterials are artificial materials that exhibit negative permittivity and permeability at terahertz (THz) frequencies, and still this class of materials is
under development. With negative values of permeability, these metamaterials can achieve the desired magnetic response and are called passive materials. This bandwidth is also known as the terahertz gap, this is because terahertz waves are electromagnetic waves with frequencies higher than microwaves but lower than infrared radiation and visible light. The terahertz frequency range used in materials research is usually defined as 0.1 to 10 THz [9].

D. Photonic Metamaterials

Photonic metamaterials are artificial material with lattice constant smaller than the wavelength of light, because of that it is hard to fabricate. For example, some metamaterials have a negative relative permittivity $\varepsilon$ and/or a negative relative permeability $\mu$. Others have a near-zero $\varepsilon$ value. Such properties lead to very peculiar wave propagation phenomena, such as negative refraction or propagation with very small phase delays. Photonic metamaterials typically contain some kind of metallic nanoscopic electromagnetic resonators, *split-ring resonators*. These can be considered as simplified LC circuits, where the inductance has been replaced with a not fully closed metallic ring and the capacitor is formed by the opening in that ring. Such metal–dielectric composites can be fabricated, e.g., with lithographic methods. When light impinges such nano-resonators, it can excite electromagnetic oscillations. These are particularly strong for frequencies near the resonance frequency, but the most interesting optical effects may occur somewhat above or below the resonance. The metamaterials with the capability of zero index of refraction (*ZIMs*) and negative values for index of refraction (*NIMs*) is the active area of research in [9]. More fundamental limitations arise from the properties of the materials used. In particular, there are no perfect conductors for frequencies of hundreds of terahertz. Therefore, particularly devices for the highest frequencies exhibit relatively high optical losses [9].
E. Tunable Metamaterials

These are the metamaterials that has the ability to randomly change the frequency dependence of the refractive index. This includes remotely controlling how an incident electromagnetic wave (EM wave) interacts with a metamaterial. This means the capability to determine whether the EM wave is transmitted, reflected, or absorbed. In general, the lattice structure of the tunable metamaterial is adjustable in real time, making it possible to reconfigure a metamaterial device during operation [16-18]. Tuning in the near infrared range is achieved by varying the permittivity of nematic liquid crystals. The metamaterials can be tuned from negative index values, to zero index or positive index values. In addition, negative index values can be increased or decreased.

F. Frequency Selective Surface (FSS) Based Metamaterials

FSS based metamaterials are the substitute to the fixed frequency metamaterials with static geometry and spacing in the unit cells used to find out the frequency response of a given metamaterials. FSS based metamaterials have option to change the frequencies in a single medium. This type of metamaterials was first developed to control the transmission and reflection characteristics of an incident radiation wave. FSS with specific geometrical shapes can be made-up as periodic arrays with elements of two dimensional planar. FSS based metamaterials has the interchangeable terminology of High Impedance Surface (HIS) or Artificial Magnetic Conductor (AMC).

Both HIS or AMC have artificial metallic electromagnetic structures. The designed structures with selection of supporting surface wave currents are different from conservative metallic conductors [9, 19].

1.4 Metamaterial Applications

A. Metamaterial Antennas

Metamaterial coatings have been used to enhance the radiation and matching properties of electrically small electric and magnetic dipole antennas. Metamaterials step up the radiated power. The newest Metamaterial antennas
radiate 95% of input radio signal at 350 MHz. Practical metamaterial antennas are as small as one fifth of a wavelength. Patch antenna with metamaterial cover have increased directivity. Flat horn antenna [9, 20] with flat aperture constructed of zero index metamaterial has the advantage of improved directivity. Because a signal propagating in a zero-index metamaterial will induce a spatially static field structure that varies in time; the phase at any point in a zero-index metamaterial will have the same constant value once steady state is reached. Metamaterials can enhance the gain and reduce the return loss of a patch antenna.

B. Metamaterial Absorber

The first Metamaterial based absorber introduced by Landy (2008) [21] utilizes three layers, two metallic layers and dielectric and shows a simulated absorptivity of 99% at 11.48 GHz as shown in Fig.(1.4). Experimentally, Landy was able to achieve an absorptivity of 88%. The difference between simulated and measured results were due to fabrication errors [22, 23].
C. Metamaterial Super Lens

Super lenses use metamaterials to go beyond the diffraction limit. Ramakrishna (2005) [24] showed it has resolution capabilities that go beyond ordinary microscopes. Conventional optical materials suffer a diffraction limit because only the propagating components are transmitted from a light source. The non-propagating components, the evanescent waves, are not transmitted. One way to improve the resolution is to increase the refractive index but it is limited by the availability of high-index materials. The road to the super lens is its aptitude to significantly enhance and recover the evanescent waves that carry information at very small scales. No lens is yet able to completely reconstitute all the evanescent waves emitted by an object. So the future challenge is to design a super lens which can constitute all evanescent waves to get perfect image [21].

D. Metamaterial Cloaks

Cloaking can be achieved by cancellation of the electric and magnetic field generated by an object or by guiding the electromagnetic wave around the object. Here guiding the wave means transforming the coordinate system in such a way that within the hollow cloak electromagnetic field will be zero. This makes the region inside the shell disappear [21, 25, 26].
E. **Metamaterial Sensor**

Metamaterials open a door for designing sensors with specified sensitivity. Metamaterials provide tools to significantly enhance the sensitivity and resolution of sensors. Metamaterial sensors are used in agriculture, biomedical etc. In agriculture the sensors are based on resonant material and employ SRR to gain better sensitivity. In bio medical wireless strain sensors, nested SRR based strain sensors have been developed to enhance the sensitivity. This is described in details by Gordon Kiti et. al (2012) [27].

F. **Metamaterial Phase Compensator**

Metamaterial act as a phase compensator. When waves passes through a (double positive slab) $DPS$ having positive phase shift and exit from a $DNG$ slab, the total phase difference is equal to zero. The concept is described by [28].
CHAPTER 2

Theory
2.1 Maxwell's Equations

Maxwell's equations are the basic to describe the electromagnetic waves, and they gather the work of Gauss, Faraday and Ampere in one theory, the theory of classical electromagnetism, after correcting Ampere's law by adding the displacement current. It was thought that electricity and magnetism were two separate forces. Until the Scottish physicist and mathematician James Clerk Maxwell published a four-part paper, "On Physical Lines of Force" between 1861 and 1862. In his paper, he formulated a set of equations that connected previously unrelated observations, experiments, and equations of electricity, magnetism, and optics into a consistent theory. They can be written in many forms (microscopic and macroscopic forms). These equations can be written in differential by [29];

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \]  \hfill (2.1)

\[ \nabla \cdot \vec{D} = \rho, \]  \hfill (2.2)

\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}, \]  \hfill (2.3)

\[ \nabla \cdot \vec{B} = 0. \]  \hfill (2.4)

where \( \vec{E} \) and \( \vec{H} \) are the electric and magnetic fields, respectively, \( \vec{D} \) and \( \vec{B} \) are the electric and magnetic flux densities, respectively, and \( \rho \) and \( \vec{J} \) are the electric charge and the current densities, respectively. For wave propagation in an isotopic medium without free charges and Conduction current, (\( \rho = 0 \), and \( \vec{J} = 0 \)), Maxwell's equations becomes:
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \]  
(2.5) 

\[ \nabla \cdot \mathbf{D} = 0, \]  
(2.6) 

\[ \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \]  
(2.7) 

\[ \nabla \cdot \mathbf{B} = 0. \]  
(2.8) 

### 2.2 Constitutive Relations

The flux densities $\mathbf{D}$, $\mathbf{B}$ and $\mathbf{J}$ are related to the $\mathbf{E}$ and $\mathbf{H}$ fields by the constitutive relations. For linear, isotropic and homogeneous media, the constitutive relation are given by

\[ \mathbf{J} = \sigma \mathbf{E} \]  
(2.9) 

\[ \mathbf{D} = \varepsilon \mathbf{E} \]  
(2.10) 

\[ \mathbf{B} = \mu \mathbf{H} \]  
(2.11) 

where the electric permittivity $\varepsilon$ and the magnetic permeability $\mu$ are defined as:

\[ \varepsilon = \varepsilon_0 \varepsilon_r(\omega) \]  
(2.12)
\[ \mu = \mu_0 \mu_r(\omega) \]  
\hspace{1cm} (2.13)

Where \( \varepsilon_r \) is the relative permittivity of the medium, \( \varepsilon_0 \) is the free space permittivity, \( \mu_r \) is its relative permeability of the medium, and \( \mu_0 \) is the free space permeability.

For charge free lossless media, where \( \sigma \) and \( J \) vanish "\( \sigma = 0, \quad J = 0 \)."

### 2.3 The Wave Equation

The solution of the wave equation describes the propagation of energy in the free charge medium. To derive the wave equation, we first take the curl of Eq. (2.5)

\[ \nabla \times \nabla \times \vec{E} = -\mu \frac{\partial \left( \nabla \times \vec{H} \right)}{\partial t} \]  
\hspace{1cm} (2.14)

Using Eq. (2.7) into Eq. (1.14)

\[ \nabla \times \nabla \times \vec{E} = \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} \]  
\hspace{1cm} (2.15)

Using the following vector identity

\[ \nabla \times \nabla \times \vec{E} = \nabla \left( \nabla \cdot \vec{E} \right) - \nabla^2 \vec{E} \]  
\hspace{1cm} (2.16)

Substituting Eq. (2.15) into Eq. (2.16) and using Eq. (2.6) yield

\[ \nabla^2 \vec{E} - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \]  
\hspace{1cm} (2.17)

Differentiation in the time domain is equivalent to multiplication by \( -i\omega \) in the frequency domain, Eq. (1.17) becomes
\[ \nabla^2 \vec{E} + k^2 \vec{E} = 0 \]  
(2.18)

Similarly, the homogenous wave equation for \( \vec{H} \) is

\[ \nabla^2 \vec{H} + k^2 \vec{H} = 0 \]  
(2.19)

where the wavenumber \( k \) is defined by \( k^2 = \omega^2\varepsilon\mu = \omega^2n^2 \).

### 2.4 Boundary Conditions

Consider an interface between two materials with a different refractive index as plotted in Fig.(2.1), and let \( \mathbf{n} \) be the unit vector in the normal direction of the interface; then, the boundary condition can be expressed as [30]:

1. The components of \( \mathbf{E} \)-field parallel to the interface are continuous across the Boundary

\[ \mathbf{n} \times (\vec{E}_1 - \vec{E}_2) = 0 \]  
(2.20)

or

\[ E_{1y} = E_{2y} \]  
(2.21)

2. The parallel components of \( \mathbf{H} \)-field are discontinuous across the boundary due to the existence of surface current density.
\[ \vec{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s \]  

(2.22)

where \( \vec{J}_s \) [A/m] represents the surface current density, in case that \( J_s = 0 \), there is

\[ \vec{n} \times (\vec{H}_1 - \vec{H}_2) = 0 \]  

(2.23)

So if there is no surface current, the tangential component of the magnetic field is continuous across the boundary, and Eq. (2.23) is equivalent to

\[ H_{1t} = H_{2t} \]  

(2.24)

(3) The component of D-field perpendicular to the interface is discontinuous if the surface charge density is nonzero

\[ \vec{n} \cdot (\vec{D}_2 - \vec{D}_1) = \rho_s \]  

(2.25)

where \( \rho_s \) [C/m\(^3\)] is the surface charge density. In case of the absence of surface charge, there is

\[ D_{1n} = D_{2n} \]  

(2.26)

which can be explained as the normal component of the \( D \) that is continuous at the boundary.

(4) The normal component of B-field at the boundary is continuous

\[ B_{1n} = B_{2n} \]  

(2.27)

2.5 Transverse Magnetic Mode (TM)

The existing mode field components are, \( H_y, E_x, E_z \), and the Helmholtz equation given by Eq.(2.19) can be modified to have the form
\[ \frac{\partial^2 H_y}{\partial x^2} + \left( k^2 - \beta^2 \right) H_y = 0 \]  

(2.28)

### 2.6 Transverse Electric Mode (TE)

The existing mode field components are \( E_y, H_x, H_z \), and Helmholtz equation given by Eq.(2.18) can be modified to have the form

\[ \frac{\partial^2 E_y}{\partial x^2} + \left( k^2 - \beta^2 \right) E_y = 0 \]  

(2.29)

### 2.7 Power Considerations

Energy flux in waveguides is an important parameter. Power fluxes inside the slab waveguide are calculated by integration of Poynting vector [31-34] averaged over the period \( T \) where:

\[ \bar{S} = \bar{E} \times \bar{H} = \frac{1}{\mu} \bar{E} \times \bar{B} \]  

(2.30)

The total power flow is given as:

\[ \bar{P} = \int \bar{S} . d \bar{A} \]  

(2.31)

Since \( P \) represents a physical quantity, it should be real. We must take the average value of \( S \) because the field is changing with time:

\[ \langle \bar{S} \rangle = \langle Re(\bar{E}) \rangle \times \langle Re(\bar{H}) \rangle \]  

(2.32)

Using the relation:

\[ \text{Re}(A) = \frac{1}{2} (A + A^*) \]  

(2.33)

we can rearrange the last equation as:
The average of the Poynting vector for time harmonic fields is:

\[ \langle \vec{S} \rangle = \frac{1}{2} [\text{Re}(\vec{E} \times \vec{H}^*)] = \frac{1}{2} [\text{Re}(\vec{E}^* \times \vec{H})] \]  

(2.34)

The average of the Poynting vector for time harmonic fields is:

\[ \overline{P}_{av} = \frac{1}{T} \int_{0}^{T} \text{Re}(\vec{E} \times \vec{H}) \, dt \]  

(2.35)

Using Maxwell’s equation and phasor notation, the last equation can be written as:

\[ \overline{P}_{av} = \frac{1}{2\omega} \text{Re} \left[ \frac{k}{\mu} |E|^2 \right] = \frac{1}{2\omega} \text{Re} \left[ \frac{k}{\varepsilon} |H|^2 \right] \]  

(2.36)

The total average Poynting vector in asymmetric three layered planer waveguides is given by:

\[ \overline{P}_{av} = \frac{1}{2\omega} \int_{-\infty}^{\infty} \text{Re} \left[ \frac{\beta}{\mu} |E|^2 \right] \, dx \]  

(2.37)

2.8 Penetration Depth

Upon propagating within a material, some energy is transmitted to atoms and electrons of that material so that the radiation is attenuated [35, 36]. For example, electromagnetic waves travel very far in dielectrics whereas they hardly penetrate in metals or semiconductors. In lossy materials such as metals, semiconductors or metamaterials, permittivity is a complex quantity so that the refractive index is also complex and takes the form of: \( \tilde{n} = n + i \kappa \). Accordingly; the complex propagation constant becomes:

\[ \tilde{k} = \frac{\omega}{c} \tilde{n}(\omega) = \frac{\omega}{c} (n + i \kappa) = k' + ik'' \]  

(2.38)

The electric field of a plane wave polarized along x-axis that travels in the x direction within such media will take the form:

\[ \vec{E} = E_0 e^{i(\vec{k}z - \omega t)} \hat{x} = E_0 e^{-i(\vec{k}z - \omega t)} e^{-i \kappa x} \]  

(2.39)

This means that the fields is exponentially attenuated upon travelling far in the medium. This exponential term is called the extinction factor. By taking the square of magnitude of both side of Eq.(2.39), the intensity, given:
\[ I = \overrightarrow{P}_{av} = I_a e^{-\alpha x} \quad (2.40) \]

\( P \) is the power flow per unit area (the magnitude of Poynting vector). The exponent \( \alpha \) is called the absorption coefficient. It takes on the form [37]:

\[ \alpha = 2k^* = 2 \frac{\omega}{c} \kappa \quad (2.41) \]

![Three-layer slab waveguide diagram](image)

**Fig. (2.2): a three-layer slab waveguide**

The measure of how long the waves will travel in the medium is called the penetration (propagation) depth. It is defined as the distance travelled along z-axis (the direction of propagation) at which the intensity is \( \frac{I_0}{e} = 0.37I_0 \). It is defined mathematically as [38]:

\[ \Delta x = \frac{1}{\alpha} = \frac{1}{2 \kappa} \frac{c}{\omega} = \frac{1}{2} \frac{1}{k_o \text{Im}(\tilde{n})} \quad (2.42) \]

We can then compute the effective of the guide width as [37]:

\[ h_{eff} = w + (\Delta x)_s + (\Delta x)_c = w + [2k_o \text{Im}(\tilde{n}_s)]^{-1} + [2k_o \text{Im}(\tilde{n}_c)]^{-1} \quad (2.43) \]

One may want to define this quantity another way that gives a good indication. Simply; this is computed directly from the effective index as:
This means that the wave appears to be confined in a wider film width. The penetration depth plays an important role in studying optical waveguide sensing because it is related to the power flow within a medium. Since sensitivity is related to fraction of power in the covering medium, thus it is directly proportional to penetration depth and inversely proportional to effective waveguide width. The fraction of power flow in a covering medium is related to the overall power as:

\[
\frac{P_i}{P} = \left[ \frac{\epsilon_f - N^2}{\epsilon_f - \epsilon_i} \right] \frac{\Delta x_i}{h_{\text{eff}}} 
\]

### 2.9 Introduction of Transfer Matrix

The 2 × 2 transfer matrix is a fruitful tool widely applied in optics and electrical engineering to treat layered systems, such as super lattices or multilayered waveguides. This approach is receiving more and more attention for its advantages such as easy computing and high accuracy. For example, M. Born and E. Wolf [29] use transfer matrix method to investigate the transmission and reflection characteristics of light propagation through multilayer structures [30-39].

When dealing with multi-lens optical devices or media, at each interface, the light is partially transmitted and partially reflected, and the matrix method provides good results [40]. This approach is a very useful, accurate and powerful technique used to study and analyse the propagation of electromagnetic waves in planer multilayer optical waveguides. When dealing with periodic layered structures (or photonic crystals), the transfer matrix method turns to have significant importance. The transfer matrix basically aims to relate the input and output of such systems.

### 2.10 Transfer Matrix General Form Satisfying Both TE and TM

In this section, we use some special solutions of the wave equation to construct a transfer matrix, which is a real matrix with clear physical insight. In the rest of this chapter, we’ll see how this method is strong, applicable and approachable.
In optics, the regions with variable refractive index are usually approximated as a series of steps at a group of points, and between two adjacent points, the refractive index is treated as constant. So the polarization transfer matrix of the TE and TM mode can be derived from the one-dimensional scalar wave equation to characterize the optical properties of these thin segments.

![Fig. (2.3), a step index planar multilayer slab waveguide](image)

In Fig. (2.3), a step index planar multilayer slab waveguide is shown. Each layer bounded by \( y - z \) planes located at \( x_{j-1} \) and \( x_j \) has width \( w_j = x_j - x_{j-1} \) and permeability \( \varepsilon_j \). The electromagnetic wave function in the \( j^{th} \) layer denoted by \( \psi_j(x_j) \) must satisfy the secular Helmholtz equation [41]:

\[
\frac{\partial^2 \psi_j(x)}{\partial x^2} + k_j^2(x) \psi_j(x) = 0
\]  

(2.46)

Where \( k_j = \sqrt{k_o^2 \varepsilon_j \mu_j - \beta^2} = \sqrt{k_o^2 n_j^2 - \beta^2} = k_o \sqrt{n_j^2 - N^2} \) represents the transverse wave number and \( \beta \) is the propagation constant. Here, \( n_j \) and \( N \) are the \( i \)th layer refractive index and the effective index, respectively. The solution of the last differential equation is well-known in literature [30]. It takes on the general form:

\[
\psi_j(x) = A_j e^{i k_j x} + B_j e^{-i k_j x}
\]  

(2.47)
Such that $A_j$ and $B_j$ are the amplitudes to be calculated from the boundary conditions. Now, the boundary conditions require that the tangential components must be continuous at the boundaries. Thus, in $TE$ modes both $\psi_j(x)$ and $-\frac{i}{\mu_j} \psi_j(x)$ must be continuous at the boundaries. Thus, in $ith$ layer, one may consider the two equations:

$$\psi_j(x) = A_j e^{ik_jx} + B_j e^{-ik_jx}$$  \hspace{1cm} (2.48)

$$\Psi'_j(x) = \frac{1}{\mu_j} \psi'_j(x) = \frac{ik_j}{\mu_j} (A_j e^{ik_jx} - B_j e^{-ik_jx})$$ \hspace{1cm} (2.49)

The last two equations can be put in a matrix form as:

$$\begin{pmatrix} \psi_j(x) \\ \Psi'_j(x) \end{pmatrix} = \begin{pmatrix} e^{ik_jx} & e^{-ik_jx} \\ -\frac{ik_j}{\mu_j} e^{ik_jx} & -\frac{ik_j}{\mu_j} e^{-ik_jx} \end{pmatrix} \begin{pmatrix} A_j \\ B_j \end{pmatrix}$$ \hspace{1cm} (2.50)

Substituting for $x = x_j$ will give:

$$\begin{pmatrix} \psi_j(x_j) \\ \Psi'_j(x_j) \end{pmatrix} = \begin{pmatrix} e^{ik_jx_j} & e^{-ik_jx_j} \\ -\frac{ik_j}{\mu_j} e^{ik_jx_j} & -\frac{ik_j}{\mu_j} e^{-ik_jx_j} \end{pmatrix} \begin{pmatrix} A_j \\ B_j \end{pmatrix}$$ \hspace{1cm} (2.51)

Upon substituting for $x = x_{j-1}$ in Eq.(2.51), then:

$$\begin{pmatrix} \psi_j(x_{j-1}) \\ \Psi'_j(x_{j-1}) \end{pmatrix} = \begin{pmatrix} e^{ik_jx_{j-1}} & e^{-ik_jx_{j-1}} \\ -\frac{ik_j}{\mu_j} e^{ik_jx_{j-1}} & -\frac{ik_j}{\mu_j} e^{-ik_jx_{j-1}} \end{pmatrix} \begin{pmatrix} A_j \\ B_j \end{pmatrix}$$ \hspace{1cm} (2.52)

Equation (2.52) can be rewritten as:
\[ (A_j) = \begin{pmatrix} e^{ik_j x_{j-1}} & e^{-ik_j x_{j-1}} \\ ik_j e^{ik_j x_{j-1}} & ik_j e^{-ik_j x_{j-1}} \end{pmatrix}^{-1} \begin{pmatrix} \psi_j(x_{j-1}) \\ \psi'_j(x_{j-1}) \end{pmatrix} \]  

(2.53)

Substituting from Eq. (2.53) into Eq. (2.51) one gets:

\[ \begin{pmatrix} \psi_j(x_j) \\ \psi'_j(x_j) \end{pmatrix} = \begin{pmatrix} e^{ik_j x_j} & e^{-ik_j x_j} \\ ik_j e^{ik_j x_j} & -ik_j e^{-ik_j x_j} \end{pmatrix} \begin{pmatrix} e^{ik_j x_{j-1}} & e^{-ik_j x_{j-1}} \\ ik_j e^{ik_j x_{j-1}} & -ik_j e^{-ik_j x_{j-1}} \end{pmatrix}^{-1} \begin{pmatrix} \psi_j(x_{j-1}) \\ \psi'_j(x_{j-1}) \end{pmatrix} \]  

(2.54)

Following some mathematics and rearranging using Maple 2016 program, we finally end up with:

\[ \begin{pmatrix} \psi_j(x_j) \\ \psi'_j(x_j) \end{pmatrix} = \begin{pmatrix} \cos(k_j w_j) & \frac{\mu_j}{k_j} \sin(k_j w_j) \\ -k_j \frac{\mu_j}{\sin(k_j w_j)} & \cos(k_j w_j) \end{pmatrix} \begin{pmatrix} \psi_j(x_{j-1}) \\ \psi'_j(x_{j-1}) \end{pmatrix} = M_j \begin{pmatrix} \psi_j(x_{j-1}) \\ \psi'_j(x_{j-1}) \end{pmatrix} \]  

(2.55)

Such that:

\[ M_j = \begin{pmatrix} \cos(k_j w_j) & \frac{\mu_j}{k_j} \sin(k_j w_j) \\ -k_j \frac{\mu_j}{\sin(k_j w_j)} & \cos(k_j w_j) \end{pmatrix} \]  

(2.56)

\[ M_j \] is the transform matrix that correlates the wave equation and its first derivative at the j'th layer boundaries where \( w_j \) represent the thickness of the core. But since \( \psi_j(x_{j-1}) = \psi_{j-1}(x_{j-1}) \) then one may write:

\[ \begin{pmatrix} \psi_j(x_j) \\ \psi'_j(x_j) \end{pmatrix} = M_j \begin{pmatrix} \psi_{j-1}(x_{j-1}) \\ \psi'_{j-1}(x_{j-1}) \end{pmatrix} \]  

(2.57)

Eq. (2.12) tells us that the wave equation at some boundary may be deduced from the wave equation at the preceding boundary with the aid of the appropriate
transformation matrix. This means that if we have \( r \) layers then the outermost layer wave equation can be connected to the first one through the general relation:

\[
\begin{pmatrix}
\psi_r(x_r) \\
\Psi'_r(x_r)
\end{pmatrix} = \prod_{j=1}^{r} M_{r-j+1} \begin{pmatrix}
\psi_1(x_1) \\
\Psi'_1(x_1)
\end{pmatrix}
\] (2.58)

When it comes to \( TM \) modes, then the two tangential components are \( \psi_j(x) \) and \( \frac{-i}{\omega \epsilon_j} \frac{\partial \psi_j(x)}{\partial x} \) must be continuous at the boundaries.

By performing out the same procedures as in \( TE \) modes, the transformation matrix in \( TM \) configuration can be proved to be:

\[
M_j = \begin{pmatrix}
\cos(k w_j) & \frac{\epsilon_j}{k_j} \sin(k w_j) \\
\frac{k_j}{\epsilon_j} \sin(k w_j) & \cos(k w_j)
\end{pmatrix}
\] (2.59)

The general form of this matrix satisfying both \( TE \) and \( TM \) configurations can be written as:

\[
M_j = \begin{pmatrix}
\cos(k w_j) & \frac{\epsilon_j^\rho \mu_j^{\rho-1}}{k_j} \sin(k w_j) \\
\frac{k_j}{\epsilon_j^\rho \mu_j^{\rho-1}} \sin(k w_j) & \cos(k w_j)
\end{pmatrix}
\] (2.60)

Such that: \( \rho = \begin{cases} 0; & TE \\ 1; & TM \end{cases} \). If the media are nonmagnetic then simply \( \epsilon_j = n_j^2 \) and the transfer matrix becomes [42]:

\[
M_j = \begin{pmatrix}
\cos(k w_j) & \frac{n_j^2 \rho}{k_j} \sin(k w_j) \\
\frac{k_j}{n_j^2 \rho} \sin(k w_j) & \cos(k w_j)
\end{pmatrix}
\] (2.61)
CHAPTER 3

Asymmetric Three Layered Planar TE Waveguides
3.1 Introduction

In this chapter, three different wave guided structures with a left-handed metamaterial are studied. This chapter focuses on a transverse electric plane polarized wave (TE) incident on the structure. The relationship between the refractive index and the frequency change in addition to the thickness of the film is studied as well as the shape of the energy in each layer. Besides that, the power flow in each layer of TE guided modes is investigated.

3.2 Theory

In this section, the dispersion relation of three layers asymmetric slab waveguide are solved using the transfer matrix method outlined in chapter 2, section 2.10.

![Diagram of a three-layer slab waveguide](image)

Fig. (3.1): Schematic diagram of a three-layer slab waveguide

The waveguide structure is given schematically in Fig.(3.1), which is a step index planar multilayer slab waveguide. A guiding layer, core, of refractive index \( n_1 \) is located within \( 0 \leq x \leq w \) and bounded from left by a semi-infinite substrate thick layer of refractive index \( n_0 \) and from the right by a cladding thick layer of refractive index \( n_2 \). All media are supposed to be dielectric nonmagnetic materials (\( \mu_r = 1 \)). The Electric field within these layers can be written as [30]:
\[
E_y = A \exp(q_0 x); \quad -\infty < x \leq 0 \quad (3.1)
\]
\[
E_y = B \cos(q_1 x)+C \sin(q_1 x); \quad 0 \leq x \leq w \quad (3.2)
\]
\[
E_y = D \exp(-q_2 (x-w)); \quad w \leq x < \infty \quad (3.3)
\]

Where \(q_0, q_1, \) and \(q_2\) are the wave numbers in substrate, core and cladding layers. The wavenumbers are given by:

\[
q_0^2 = k_0^2 \left(N^2 - \varepsilon_0 \mu_0\right)
\]
\[
q_1^2 = k_0^2 \left(\varepsilon_i \mu_i - N^2\right)
\]
\[
q_2^2 = k_0^2 \left(N^2 - \varepsilon_2 \mu_2\right)
\]

According to Eq.(2.60) in Chapter 2, we have:

\[
\begin{pmatrix}
E_1(w) \\
E'_1(w)
\end{pmatrix} =
\begin{pmatrix}
E_2(w) \\
E'_2(w)
\end{pmatrix} =
\begin{pmatrix}
\cos(q_1 w) & \frac{\mu_j}{q_1} \sin(q_1 w) \\
-\frac{q_1}{\mu_j} \sin(q_1 w) & \cos(q_1 w)
\end{pmatrix}
\begin{pmatrix}
E_0(0) \\
E'_0(0)
\end{pmatrix} \quad (3.4)
\]

Substituting for the fields in the last equation gives:

\[
\begin{pmatrix}
D \\
-q_2 D
\end{pmatrix} =
\begin{pmatrix}
\cos(q_1 w) & \frac{\mu_j}{q_1} \sin(q_1 w) \\
-\frac{q_1}{\mu_j} \sin(q_1 w) & \cos(q_1 w)
\end{pmatrix}
\begin{pmatrix}
A \\
q_0 A
\end{pmatrix} \quad (3.5)
\]

Or simply:

\[
\begin{pmatrix}
\cos(q_1 w) & \frac{\mu_j}{q_1} \sin(q_1 w) \\
-\frac{q_1}{\mu_j} \sin(q_1 w) & \cos(q_1 w)
\end{pmatrix}
\begin{pmatrix}
A \\
q_0 A
\end{pmatrix} -
\begin{pmatrix}
D \\
-q_2 D
\end{pmatrix} = 0 \quad (3.6)
\]

Solving Eq.(3.6), we get the two following equations:
\[ A \cos(q \omega t) + q \varphi A \frac{\mu_j}{q_1} \sin(q \omega t) - D = 0 \quad (3.7) \]

\[-A q_1 \sin(q \omega t) + q_\varphi A \cos(q \omega t) + q_z D = 0 \quad (3.8)\]

Eliminating A and D from equations (3.7 and 3.8), one ends with:

\[ q_0 \cos(q \omega t) + q_\varphi q_2 \frac{\mu_j}{q_1} \sin(q \omega t) - \frac{q_1}{\mu_j} \sin(q \omega t) + q_2 \cos(q \omega t) = 0 \quad (3.9) \]

Dividing Eq.(3.9) by \( \cos(q_1 \omega t) \) and arranging we get:

\[ \tan(q_1 \omega t) = \frac{1}{q_1} \frac{q_0 + q_2}{\frac{1}{\mu} - \frac{\mu_j}{q_1^2} q_0 q_2} \quad (3.10) \]

Making use of the relation: \( \tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \),

We can rearrange Eq.(3.10) as:

\[ \tan(q_1 \omega t) = \tan(\tan^{-1} \frac{q_0 \mu_j}{q_1} + \tan^{-1} \frac{q_2 \mu_j}{q_1}) \quad (3.11) \]

which is simplified to the normal well-known dispersion relation [30, 43]:

\[ q \omega t = \tan^{-1} \frac{q_0 \mu_j}{q_1} + \tan^{-1} \frac{q_2 \mu_j}{q_1} + m \pi; \quad m = 0, 1, 2, ... \quad (3.12) \]

### 3.3 Power flow in three layers

The time-averaged power flowing in the substrate, the film and the cladding layers are respectively given by Eq.(2.41) [44]:

\[ P_0 = \frac{1}{2 \omega} \int_{-\infty}^{0} \text{Re} \left[ \frac{\beta}{\mu} A e^{q_0 x} \right]^2 dx = \frac{1}{4 \omega \mu_0 q_0} \beta A^2 \quad (3.13) \]
\[
P_1 = \frac{1}{2\omega} \int_0^w \Re \left[ \frac{\beta}{\mu} B \cos(q_x w) \right]^2 dx = \frac{1}{4} \frac{\beta B^2 (w + \sin(q_1 w) \cos(-q_1 w + 2a_1))}{\omega \mu_i} \quad (3.14)
\]

\[
P_2 = \frac{1}{2\omega} \int_0^w \Re \left[ \frac{\beta}{\mu} C e^{(-q_2(x-w))} \right]^2 dx = \frac{1}{4} \frac{\beta C^2}{\omega \mu_2 q_2} \quad (3.15)
\]

From the boundary conditions, the amplitudes \( A, B \) and \( C \) can be given (as in section 2.4) by applying the boundary conditions when \( x = 0 \) and \( x = w \), then we can find the value of \( C \) and \( B \) as,

\[
C = \frac{A \cos(-q_1 w + a_i)}{\cos(a_i)} \quad (3.16)
\]

\[
B = \frac{A}{\cos(a_i)}
\]

where \( a_i \) represents the phase angle difference which can be written as:

\[
a_i = \tan^{-1}\left(\frac{q_0}{q_i}\right) \quad (3.17)
\]

The total power flowing in the waveguide structure can be obtained by summing the power in a three layers as

\[
P_{\text{total}} = P_{\text{core}} + P_{\text{cladding}} + P_{\text{substrate}} \quad (3.18)
\]

The power confinement factor in the core is defined and the power flow in the core to the total power flow in the waveguide structure [45]:

\[
\Gamma = \frac{P_{\text{core}}}{P_{\text{total}}} \quad (3.19)
\]
3.4 Numerical Analysis and Discussion

3.4.1: Structures Comprising LHM Core:

![Diagram of three-layer structure](image)

The structure under consideration consists of an asymmetric slab waveguide with LHM layer occupying the region $0 < x < w$, with permittivity $\varepsilon_1$ and magnetic permeability $\mu_1$ are simultaneously negative so that the refractive index takes on negative values at predetermined frequency. In microwave band, the frequency dependent permittivity is described by the Drude medium model as [46-48]:

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i\gamma \omega}$$  \hspace{1cm} (3.20)

and

$$\mu(\omega) = 1 - \frac{F \omega_p^2}{\omega^2 - \omega_0^2 + i\gamma \omega}$$  \hspace{1cm} (3.21)

Where $\omega_p$ is plasma frequency, $\omega_0$, the resonance frequency, $\gamma$, electron scattering, and $F$ is fractional area of unit cell occupied by the split ring [49].

Using Maple Software, a computer code is developed to solve the dispersion relation numerically to compute the effective refractive index, and then the dispersion curves could be plotted. The code is also designed to find the electric field profiles in waveguide structure, for various configurations, and finally we could
calculate the power flow within the structure versus different waveguide parameters. The LHM core is characterized by the following: \( \omega_p = 1.2 \times 10^{16}, \gamma = 1.2 \times 10^{14} \) and \( F = 0.56 \) [46]. Left handed metamaterials in terahertz frequencies are real hard to fabricate [50]. Thus, we have used an ad-hoc model to mimic the behavior of terahertz left handed metamaterial with the following electric properties: \( \mu_1 = -1 - 0.004i \), and for different values of the core permittivity. In this analysis, permittivity and permeability for the cover and substrate layers respectively are: for non-dispersive cladding \( \varepsilon_0 = 2.99 \), and non-dispersive substrate is \( \varepsilon_2 = 3.38 \). The two media are considered nonmagnetic and non-dispersive so that \( \mu_0 = \mu_3 = 1 \).

In Fig.(3.3), we plot the real part of effective refractive index versus the different mode order. It can be seen from Fig.(3.3) that the real refractive index for
the fundamental mode $TE_0$ increase as the frequency increases, and claims negative index values which means the structure behaves like negative index material. Meanwhile, the other higher order modes, $TE_1$ and $TE_2$, decreases with increasing frequency, and they claim positive index, thus the structure behaves like right handed material, or positive index material. This means that the group velocity is positive for $TE_0$ (forward waves) and negative for other modes (backward waves) [51]. The effective refractive index is shown to be higher valued in higher modes. Accordingly, since the effective index is high the structure can be used in all optical circuit as time-delay element to create synchronization between optical branches [52-53].

![Dispersion curves: Imaginary refractive index against frequency for: a) TE$_0$ (solid line), b) TE$_1$ (dashed) and c) TE$_2$ (dotted), Thickness $w = 200nm$, $\varepsilon_0 = 2.99$, $\mu_1 = -1 - 0.004i$, $\varepsilon_2 = 3.38$, $\omega_p = 1.2 \times 10^{16}\frac{rad}{s}$, $\omega_0 = 1.2 \times 10^{14}\frac{rad}{s}$, $F = 0.56$.](image)

In Fig.(3.4), we study the absorption properties of the structure. The relation between the imaginary part of the modal effective index, the imaginary part, $Im(N)$,
and frequency for different mode orders is studied. We examine the first three modes, namely $TE_0$, $TE_1$, and $TE_2$, for the same parameters used to obtain Fig.(3.3). It is clear that the absorption or attenuation for the fundamental mode $TE_0$ decreases as frequency increases, meanwhile the attenuation increase with the frequency increase for higher modes. It is clear from the figure that the absorption is high this structure. Since we have high effective refractive index in the structures means that the propagation is slow, which gives more opportunity for absorption. Consequently, this structure can be used as attenuator for strong signals besides time delay element, but is not suitable for designing of long waveguides.

![Dispersion curves](image)

**Fig.(3.5)**: Dispersion curves: Real refractive index against frequency for different thicknesses: a) $w = 20 \text{ nm}$ (solid line), b) $w = 40 \text{ nm}$ (dashed) and c) $w = 60 \text{ nm}$ (dotted), for $TE_0$, mode., $\varepsilon_0 = 2.99, \mu_1 = -1 - 0.004i, \varepsilon_2 = 3.38, \omega_p = 1.2 \times 10^{16} \frac{\text{rad}}{\text{T}}, \omega_0 = 1.2 \times 10^{14} \frac{\text{rad}}{\text{T}}, F = 0.56$.

From Fig.(3.5), it can be seen that as the thickness increases the effective refractive index decreases for $TE_0$ mode. This means that the group velocity is negative. The
effective refractive index is shown to be higher valued in higher frequencies, in the end the general behavior of the form is \textit{LHM} [54].

![Graph](image)

Fig.(3.6): Dispersion curves: Imaginary refractive index against frequency for different thicknesses: a) \(w = 20 \text{ nm} \) (solid line), b) \(w = 40 \text{ nm} \) (dashed) and c) \(w = 60 \text{ nm} \) (dotted), for TE_0 mode., \( \varepsilon_0 = 2.99, \mu_1 = -1 - 0.004i, \varepsilon_2 = 3.38, \omega_p = 1.2 \times 10^{16} \text{ rad} \), \( \omega_0 = 1.2 \times 10^{14} \text{ rad} \), \( F = 0.56 \).

To study the absorption properties of the structure, the relation between the imaginary part of the modal effective index and thickness is carried out in Fig.(3.6). This is expected to shed light on the extinction or absorption coefficient. It is clear that the imaginary part of the refractive index decreases as thickness increases. That is, absorption decreases with frequency which is reasonable. It is clear that, the change of the imaginary part refractive index is very small and decreases as thickness increasing. That means we have lower loss structure.
Fig.(3.7) Modal field distribution of the structure under consideration for TE$_0$. Thickness $w = 200nm$, $\varepsilon_0 = 2.99, \mu_1 = -1 - 0.004d, \varepsilon_2 = 3.38, \omega_p = 1.2 \times 10^{16} \frac{rad}{s}, \omega_0 = 1.2 \times 10^{14} \frac{rad}{s}, F = 0.56$.

The plot shows the modal field distribution of the proposed waveguide in Fig.(3.7) for the same previous parameters given in Fig.(3.3) is used. The field distributions are in expected shape for the fundamental mode ($m = 0$). As seen from figure, the electromagnetic field consists of one peak trapped in the core, which means that the electromagnetic waves are collected, confined, in the core. Fig.(3.7) also shows that the tails represent the distribution of an evanescent electric field in substrate and cladding.
Comparing the electric profile for $TE_0$, and $TE_1$, mode. It is clear that the peak of curve is decreasing as the number of mode increasing. As we see from Fig.(3.8), when $m = 1$ there are two peaks for electric field in the film. Fig.(3.8) also shows that there are two tails at the end of profile which represent an evanescent electric field in substrate and cladding, which means no energy flow in substrate and cladding layers and thus the value of Poynting vector is zero there.
Comparing the electric profiles for $TE_0$, $TE_1$, and $TE_2$ modes. From Fig.(3.9) it is clear that the number of peaks increases as the number of mode increases. In summary, the electric field amplitude decreases as the mode number increases. Moreover, the electric field is oscillatory in the guiding layer and evanescent tails in the substrate and cladding layers. This structure shows good guidance for the $TE$ waves.
Fig.(3.10): Power flow in a three layers waveguide structures comprising LHM core

Fig.(3.10.a) illustrates the amount of power flowing in the three layers. It is evident, that the power in the guiding layer near the cladding layer appears to be positive and this indicates that the behavior is a $RHM$, but on the other hand, the power appears negative in the guide layer near the substrate layer and this indicates that the behavior become $LHM$. It is worth to notice that the power is confined within the core layer, while the rest is a very small amount distributed over both substrate and cladding layers, as shown in Fig.(3.10.b) and (3.10.c). Therefore, this
structure is good for transmitting signals for short distances, since the extinction coefficient is considerably high. One of the most likely applications is in all optical circuits.

![Graph showing penetration depth (Δx) vs. frequency (THz) for different core thicknesses: 1) w = 100 nm (solid line), 2) w = 300 nm (dashed) and 3) w = 500 nm (dotted). For TE₀ mode, ε₀ = 2.99, μ₁ = −1 − 0.004i, ε₂ = 3.38, ωₚ = 1.2 × 10^{16} \ rad/\ s, ω₀ = 1.2 × 10^{34} \ rad/\ s, F = 0.56.]

In Fig.(3.11), we study the penetration depth [38] of the structure to a measure of how deep light or any electromagnetic radiation can penetrate into a material. The relation between the penetration depth (Δx) and frequency for different thicknesses is studied. Fig.(3.11) shows that penetration depth increases gradually with increasing frequency. Moreover, the penetration depth increases with core thickness increase.
Fig.(3.12): Effective guide ($h_{eff}$) for different of thicknesses: 1) $w = 100 \, \text{nm}$ (solid line), 2) $w = 300 \, \text{nm}$ (dashed) and 3) $w = 500 \, \text{nm}$ (dotted), for $TE_0$ mode. $\varepsilon_0 = 2.99, \mu_1 = -1 - 0.004i, \varepsilon_2 = 3.38, \omega_p = 1.2 \times 10^{16} \, \text{rad} \, \text{s}^{-1}, \omega_0 = 1.2 \times 10^{14} \, \text{rad} \, \text{s}^{-1}, F = 0.56$.

Fig.(3.12) shows the relation between the effective guide ($h_{eff}$) and the operating frequency, for different thicknesses. We notice that, the effective guide increases as the frequency increases, and the ($h_{eff}$) increases with increasing core thickness increase.
Fig.(3.13): Power curves for different thicknesses: 1) $w = 100 \text{ nm}$ (solid line), 2) $w = 300 \text{ nm}$ (dashed) and 3) $w = 500 \text{ nm}$ (dotted), for $TE_0$ mode, $\varepsilon_0 = 2.99$, $\mu_1 = -1 - 0.004i$, $\varepsilon_2 = 3.38$, $\omega_p = 1.2 \times 10^{16} \text{ rad s}^{-1}$, $\omega_0 = 1.2 \times 10^{14} \text{ rad s}^{-1}$, $F = 0.56$.

It is important to study the sensitivity, which is related to fraction of power in the covering medium. In Fig.(3.13.a), we plot the fraction of power flow in cladding for $TE_0$ mode versus the allowed frequency range for different core’s thickness. This shows that the fraction of power flow increases as the frequency increases. From Fig.(3.13.b), it can be seen the fraction of power in substrate increases as increasing the frequency. It is clear that from Fig.(3.13.a) and Fig.(3.13.b), the power in cladding and substrate increases as the thickness increasing, thus the structure has the potential to be used as a sensor.

In Fig.(3.14), we plot the confinement factor for $TE_0$ mode over the specified range of frequencies. The structure high confinement for the specified frequency range, and it touch 100% percent at $1.2 \text{ THz}$. Thus, this structure is a very good candidate for wave guiding in all optical circuits, since we need very tight guiding with reasonable loss.
Fig(3.14): Power confinement factor for $TE_0$ mode, core thickness is Thickness $w = 200\text{nm}$

3.4.2: Structures Comprising ENG Substrate:

![Diagram of a three-layers slab waveguide structure including ENG substrate]

In this case, the dispersion properties of the proposed structure are studied when the core and cladding are $RHM$ and the substrate is ENG material with negative permittivity. In discussion, however, the $ENG$ medium is chosen to have:
\( \varepsilon_2 = -2.6 + 0.4I, \mu_2 = 1 - 0.004I. \) Permeability and Permittivity for the other layers are generally given as: \( \varepsilon_0 = 2.99, \varepsilon_1 = 3.38 \) and the media are considered nonmagnetic so that \( \mu_0 = \mu_1 = 1. \)

In Fig.(3.16), the dispersion curves for different values of modes are plotted in frequency range from 3.4 to 3.5 terahertz and effective refractive index when the thickness is fixed at 800nm. There are no solutions when the frequency less than 3.4 terahertz and greater than 3.5 terahertz. We notice that real refractive index increases with increasing frequency with positive slope which means that the group velocity become faster, obviously the behavior is \textit{RHM}. 

\[ \text{Fig.(3.16) : Dispersion curves: Real refractive index against frequency for: a) TE}_0 \text{ (solid line), b) TE}_1 \text{ (dashed) and c) } \text{TE}_2 \text{. Thickness } w = 800\text{nm}., \varepsilon_0 = 2.99, \mu_2 = 1 - 0.004I, \varepsilon_1 = 3.38, \omega_p = 1.2 \times 10^{16} \frac{\text{rad}}{\text{m}}, \omega_0 = 1.2 \times 10^{14} \frac{\text{rad}}{\text{m}}, F = 0.56. \]
Fig. (3.17): Dispersion curves: Real refractive index against frequency for different thicknesses:

a) $w = 800 \text{ nm}$ (solid line), b) $w = 850 \text{ nm}$ (dashed) and c) $w = 900 \text{ nm}$ (dotted), for TE$_0$ mode. $\varepsilon_0 = 2.99, \mu_2 = 1 - 0.004I, \varepsilon_1 = 3.38, \omega_p = 1.2 \times 10^{16} \text{ rad/s, } \omega_0 = 1.2 \times 10^{14} \text{ rad/s, } F = 0.56.$

From Fig. (3.17), it can be seen that as the thickness increases, the effective refractive index increases at TE$_0$. This means that the group velocity is positive. The effective refractive index is shown to be higher valued in higher thickness. Fig. (3.17) illustrates the change of the real part refractive index is very small and increases with increasing the thickness with positive slope. That means the group velocity is positive, in the end the general behavior of the form is RHM[54].
Fig.(3.18) : Dispersion curves: Imaginary refractive index against frequency for different thicknesses: a) $w = 800 \text{ nm}$ (solid line), b) $w = 850 \text{ nm}$ (dashed) and c) $w = 900 \text{ nm}$ (dotted), for TE$_0$ mode. $\varepsilon_0 = 2.99$, $\mu_2 = 1 - 0.004 I$, $\varepsilon_1 = 3.38$, $\omega_p = 1.2 \times 10^{16} \frac{\text{rad}}{\text{sec}}$, $\omega_0 = 1.2 \times 10^{14} \frac{\text{rad}}{\text{sec}}$, $F = 0.56$.

The absorption characteristics of the waveguide are important factors in determining the wavelength used. The absorption properties of the structure are studied through the relation between the imaginary part of the modal effective index and thickness is carried out in Fig.(3.18). It is clear that the imaginary part of the refractive index increases as frequency increases. That is, absorption increases with frequency, which is reasonable because the real refractive index is low. Again, the small change in the change of the imaginary part means that we have low loss structure. Also, the absorption increase as the core thickness increase.
We can notice that, the electromagnetic field is trapped in the core, which means that the $EM$ waves are confined in the core. Comparing the electric profile for $TE_0$, $TE_1$ modes. It is clear that the number of peaks of curve increasing as the number of mode increasing. This combination indicates that it is a good guidance and an evanescent electric field in substrate and cladding can be used in an application such as optical sensors. The fields away from the film die off exponentially and those fields are thus described as evanescent electric field in substrate and cladding in the $x$ direction.

The evanescent wave from an optical fiber can be used in a gas sensor, it has been demonstrated that fiber-optic evanescent waves can be used for absorption gas spectroscopy [55].
Fig. (3.20.a) represents the amount of power flowing in the three layers. It can be seen the power in the guiding layer near the cladding layer appears to be negative and this indicates that the behavior is a *ENG* material but on the other hand, the power appears positive in the guide layer near the substrate layer and this indicates
that the behavior become $ \text{RHM} $. It is also noticeable that the power is confined to the guiding layer, while the rest is a very small amount distributed on both substrate and cladding. Fig. (3.20.c) shows the power in cladding layer, which increases slightly and then returns to zero. We can conclude that this structure is good for waveguide for long distance because the lost energy will not be great.

![Graph showing penetration depth (\( \Delta x \)) for different frequencies for different thicknesses: (a) \( w = 900 \, \text{nm} \) (solid line), (b) \( w = 940 \, \text{nm} \) (dashed) and (c) \( w = 980 \, \text{nm} \) (dotted), for TE\(_0\) mode, \( \varepsilon_0 = 2.99, \mu_2 = 1 - 0.004i, \varepsilon_1 = 3.38, \omega_p = 1.2 \times 10^{16} \, \text{rad/s}, \omega_0 = 1.2 \times 10^{14} \, \text{rad/s}, F = 0.56 \).](image)

In Fig. (3.21), The relation between the Penetration depth (\( \Delta x \)) and frequency for different thickness is studied. From Fig. (3.21), it can be seen the penetration depth increases gradually with increasing frequency. As well as through the Fig. (3.21), it is clear that, the penetration depth decreases with increasing thickness.
Fig. (3.2) shows the relation between the effective guide \( h_{\text{eff}} \) and the operating frequency from 3.5 to 4 terahertz, for different values of thickness. It can be seen that, the effective guide \( h_{\text{eff}} \) increases as the frequency increases. From Fig. (3.2), it's clear that, the effective guide decreases as the thickness increasing.
Fig. (3.23) : Power curves for different thicknesses: a) $w = 900 \text{ nm}$ (solid line), b) $w = 940 \text{ nm}$ (dashed) and c) $w = 980 \text{ nm}$ (dotted), for $TE_0$ mode, $\varepsilon_0 = 2.99, \mu_2 = 1 - 0.004 I, \varepsilon_1 = 3.38, \omega_p = 1.2 \times 10^{16} \frac{\text{rad}}{\text{s}}, \omega_0 = 1.2 \times 10^{14} \frac{\text{rad}}{\text{s}}, F = 0.56$.

In Fig. (3.23.a), we plot the fraction of power flow in cladding for $TE_0$ mode versus the allowed frequency range for different core thickness. This shows that the fraction of power flow decreases as the frequency increases, and also, the power increase with core thickness increase. From Fig. (3.23.b), it can be seen the fraction of power in substrate decreases sharply as increasing the frequency, meanwhile, the core thickness effect is small.

Confinement factor is plot in Fig. (3.24) for $TE_0$ mode over the specified range of frequencies. The structure shows high confinement and slowly decreases with frequency increase. By the same token, the structure is a very good candidate for wave guiding in all optical circuits.
3.4.3: Structures Comprising ENG Cladding:

In this case, the dispersion properties of the proposed structure are studied when the core and substrate are *RHM* and the cladding is epsilon negative material with
negative permittivity. We consider a left handed material cladding with \( \varepsilon_0 = -2.5 + 0.39i, \mu_0 = 1 - 0.004i \). Dielectric permittivity and magnetic permeability for the core and substrate are generally given as: \( \varepsilon_1 = 2.99, \varepsilon_2 = 3.38 \) and the media are considered nonmagnetic so that \( \mu_1 = \mu_2 = 1 \).

It is significant to study the propagation of waves in a three-layers slab waveguide containing epsilon negative materials in cladding. Fig.(3.26) shows the relation between the real refractive index and the operating frequency from 3.4 to 3.5 terahertz, for different values of thickness. For this structure, we found a solution in a narrow frequency range, between 3.4 to 3.5 terahertz, and there is no solution higher mode order, for \( m = [1,2,3] \). This indicates that this combination supports only one mode, which means one mode carries all energy. We notice that, real refractive index

![Dispersion curves: Real refractive index against frequency for different thicknesses: (a)w = 800 nm (solid line), (b) w = 850 nm (dashed) and (c) w = 900 nm (dotted), for TE0 mode, \( \varepsilon_1 = 2.99, \mu_0 = 1 - 0.004i, \varepsilon_2 = 3.38, \omega_p = 1.2 \times 10^{16} \text{rad} s^{-1}, \omega_0 = 1.2 \times 10^{14} \text{rad} s^{-1}, F = 0.56 \).](image)
decreases as the frequency increases with negative group velocity, that means the group velocity is negative [45].

![Dispersion curves: Imaginary refractive index against frequency for different thicknesses: a) w = 800 nm (solid line), b) w = 850 nm (dashed) and c) w = 900 nm (dotted), for TE₀ mode, ε₁ = 2.99, µ₀ = 1 − 0.0044l, ε₂ = 3.38, ωₚ = 1.2 × 10^{16}, ω₀ = 1.2 × 10^{14}, F = 0.56.](image)

To study the absorption properties of the structure, the relation between the imaginary part of the modal effective index and frequency is carried out in Fig.(3.27).
As mentioned above, absorption increases with both frequency and thickness. Being very small means that the structure losses are low.
Fig.(3.28) Modal field distribution of the structure under consideration for TE_{0}, Thickness \( w = 850\, \text{nm} \), \( \varepsilon_1 = 2.99 \), \( \mu_0 = 1 - 0.004i \), \( \varepsilon_2 = 3.38 \), \( \omega_p = 1.2 \times 10^{16} \, \text{rad} / \text{s} \), \( \omega_0 = 1.2 \times 10^{14} \, \text{rad} / \text{s} \), \( F = 0.56 \).

The electric field profile for the fundamental mode \( (m = 0) \) is shown in Fig.(3.28) for the same previous parameters given in Fig.(3.28). The field distributions are in expected shape for the fundamental mode \( (m = 0) \). As seen from figure, the electromagnetic field consists of one peak trapped in the core, which means that the electromagnetic waves are collected and confined in the core. Fig.(3.28) also shows that the tails represent the distribution of an evanescent electric field in substrate and weak radiating mode in the cladding. Moreover, this combination at the above values only supports fundamental mode \( (m = 0) \) and does not support higher modes.
It is important to know how much power is flowing in the combination. Fig.(3.29.a) represents the amount of power flowing in the three layers. The power in the guiding layer near the cladding layer appears to be negative and this indicates that the behavior is a ENG material but on the other hand, the power appears positive in the guide layer near the substrate layer and this indicates that the behavior become
It is also noticeable that the power is confined to the guiding layer, it is therefore a good combination of optical transport and we conclude this structure is good for waveguide for long distance because the lost energy is small.

**Fig.(3.30)**: Penetration depth ($\Delta x$) for different thicknesses: a) $w = 800 \text{ nm}$ (solid line), b) $w = 840 \text{ nm}$ (dashed) and c) $w = 880 \text{ nm}$ (dotted), for $TE_0$ mode, $\varepsilon_1 = 2.99$, $\mu_0 = 1 - 0.004i$, $\varepsilon_2 = 3.38$, $\omega_p = 1.2 \times 10^{16} \frac{\text{rad}}{\text{s}}$, $\omega_0 = 1.2 \times 10^{14} \frac{\text{rad}}{\text{s}}$, $F = 0.56$.

Penetration depth is a term that describes the decay of electromagnetic waves inside of a structure. In Fig.(3.30), we plot the penetration depth for $TE_0$ mode versus the allowed frequency range for different core’s thicknesses. Fig.(3.30) shows that the penetration depth decreases with core thickness increase. From Fig.(3.30), it can be seen the penetration depth decreases gradually with frequency increase.
Fig. (3.31) shows the relation between the effective guide \( h_{eff} \) and the operating frequency from 3.5 to 3.8 terahertz for different values of thicknesses. It can be noticed that the effective guide decreases as the frequency increases, and also the effective guide thickness decrease with frequency increase for all core thicknesses.
In Fig.(3.32.a), we plot the fraction of power flow in cladding for $TE_0$ mode versus the allowed frequency range for different core thicknesses. Fig.(3.32.a) shows that the fraction of power flow increases as the frequency increases. It is clear that, the fractional power flow doesn’t change appreciably as the core thickness increases.

From Fig.(3.32.b), it can be seen that the fraction of power in substrate decreases with increasing the frequency. It is clear that from Fig.(3.32.b), the power in substrate decreases with increasing core thickness.

Finally, in Fig.(3.33), we plot power confinement factor versus the operating frequency. Power confinement factor increases with frequency increase. Here, we have an interesting structure with low effective refractive index, low power loss (small extinction coefficient), and supports only one mode (the incident power will be carried by one mode only, under careful power coupling), but has low power confinement.
Fig.(3.33): Power confinement factor for $TE_0$ mode, core thickness is $\text{Thickness } w = 850\text{nm}$
CHAPTER 4

TE Mode In Asymmetric Four Layered Planar Waveguides
4.1 Introduction

In this chapter, two different waveguide structures with a left-handed metamaterial are studied. The propagation of a transverse electric plane polarized wave (TE) incidence on the structure is put into consideration. The influence of both frequency and thickness of the inner slabs on the modal effective index are also studied between the refractive index and the frequency change is studied. In addition, the electric filed distribution and the power flow in each layer are derived and plotted.

4.2 Theory

As shown in Fig.(4.1), a guiding film of refractive index \( n_1 \) is sandwiched between a substrate with refractive index \( n_0 \) and another layer of \( n_2 \) as refractive index. The whole arrangement is covered from above by a semi-infinite clad of refractive index \( n_3 \). The refractive indices are chosen such that \( n_0 < n_3 < N \) and \( N < n_2 < n_1 \). That is, the electromagnetic fields decay in layer 0 and 3, and oscillate in layer 1 and 2. The electromagnetic waves are assumed to be propagating along the \( z \)-axis so that the solution of Helmholtz' equation is written as:

![Fig(4.1): Four layer slab waveguide](image-url)
\[ \psi_x = \begin{cases} 
A \exp(\gamma_0 x) ; & -\infty < x \leq 0 \\
B \exp(i \gamma_1 x) + C \exp(-i \gamma_1 x) ; & 0 \leq x \leq h_1 \\
D \exp[i \gamma_2 (x - h_1)] + E \exp[-i \gamma_2 (x - h_1)] ; & h_1 \leq x \leq h_2 \\
F \exp(-\gamma_3 (x - h_1 - h_2)) ; & h_1 + h_2 \leq x < \infty 
\end{cases} \tag{4.1} \]

Where \( \gamma_i = \frac{q_i}{\mu_i} \), we can write the electric field equations as:

\[
\begin{align*}
E_0 &= A_0 \exp(\gamma_0 x) \\
E_1 &= A_1 \cos(\gamma_1 x - a_1) \\
E_2 &= A_2 \cos(\gamma_2 (x - h_1) + a_2) \\
E_3 &= A_3 \exp(-\gamma_3 (x - h_1 - h_2))
\end{align*} \tag{4.2}
\]

where \( a_1 \) and \( a_2 \) are phase differences which can be written as,

\[
\begin{align*}
a_1 &= \arctan \left( \frac{\gamma_0}{\gamma_1} \right) \\
a_2 &= \arctan \left( \frac{\gamma_2}{\gamma_3} \right) \tag{4.3}
\end{align*}
\]

where \( A_0, A_1, A_2 \) and \( A_3 \) are the wave amplitudes in the four layers that can be determined from the boundary conditions. The parameters \( q_0, q_1, q_2 \) and \( q_3 \) are given by:

\[
\begin{align*}
\gamma_0^2 &= k_0^2 \left( N^2 - \varepsilon_0 \mu_0 \right) \\
\gamma_1^2 &= k_0^2 \left( \varepsilon_1 \mu_1 - N^2 \right) \tag{4.4} \\
\gamma_2^2 &= k_0^2 \left( \varepsilon_2 \mu_2 - N^2 \right) \\
\gamma_3^2 &= k_0^2 \left( N^2 - \varepsilon_3 \mu_3 \right)
\end{align*}
\]

and

\[
\gamma_1^2 = k_0^2 \left( \varepsilon_1 \mu_1 - N^2 \right) \tag{4.5}
\]
Applying the boundary conditions at the three interfaces \((x = 0, x = h_1 \text{ and } h_1 + h_2)\), one can write:

Since \(E_0 = E_1\) at \(x = 0\) then:

\[
A_0 = A_1 \cos(a_1)
\]  
(4.6)

Again, since \(E_1 = E_2\) at \(x = h_1\) then:

\[
A_1 \cos(-h_1 \gamma_1 + a_1) = A_2 \cos(a_2)
\]  
(4.7)

From Eq(4.7) the amplitude in the third layer is:

\[
A_2 = \frac{A_1 \cos(-h_1 \gamma_1 + a_1)}{\cos(a_2)}
\]  
(4.8)

At \(x = h_1 + h_2\), \(E_2 = E_3\), then:

\[
A_3 = A_2 \cos(h_2 \gamma_2 + a_2)
\]  
(4.9)

Substituting for \(A_2\) from Eq.(4.8) into Eq.(4.9) and using some mathematics, the result is:

\[
E_0 = A \cos(a_1) \exp(\gamma_0 x)
\]

\[
E_1 = \frac{A \cos(-h_1 \gamma_1 + a_1) \cos(\gamma_2 (x - h_1) + a_2)}{\cos(a_2)}
\]  
(4.10)

\[
E_2 = AB \cos(\gamma_2 (x - h_1) + a_2)
\]

\[
E_3 = AB \cos(h_2 \gamma_2 + a_2) \exp(-\gamma_3 (x - h_1 - h_2))
\]

where \(B\) is:

\[
B = \frac{\cos(-h_1 \gamma_1 + a_1)}{\cos(a_2)}
\]  
(4.11)

Again, applying Eq.(2.55) in Chapter 2, we can have:
\[
\begin{pmatrix}
F \\
-\gamma_3 F
\end{pmatrix} = M_1 M_2 \begin{pmatrix} A \\ \gamma_0 A \end{pmatrix}
\]  \hspace{1cm} (4.12)

with:

\[
M_1 = \begin{pmatrix}
\cos(\gamma_1 h_i) & \frac{1}{\gamma_1} \sin(\gamma_1 h_i) \\
-\gamma_1 \sin(\gamma_1 h_i) & \cos(\gamma_1 h_i)
\end{pmatrix}
\]  \hspace{1cm} (4.13)

and

\[
M_2 = \begin{pmatrix}
\cos(\gamma_2 h_2) & \frac{1}{\gamma_2} \sin(\gamma_2 h_2) \\
-\gamma_2 \sin(\gamma_2 h_2) & \cos(\gamma_2 h_2)
\end{pmatrix}
\]  \hspace{1cm} (4.14)

The overall transfer matrix (i.e. the product \(M = M_1 M_2\)) can be given as:

\[
M = \cos(\gamma_2 h_2) \cdot \cos(\gamma_1 h_1) \begin{pmatrix}
1 - \gamma_1 \frac{1}{\gamma_2} \tan(\gamma_2 h_2) \cdot \tan(\gamma_1 h_1) & \frac{1}{\gamma_1} \tan(\gamma_1 h_1) + \frac{1}{\gamma_2} \tan(\gamma_2 h_2) \\
-\gamma_1 \tan(\gamma_2 h_2) - \gamma_1 \tan(\gamma_1 h_1) & 1 - \frac{1}{\gamma_1} \gamma_2 \tan(\gamma_2 h_2) \cdot \tan(\gamma_1 h_1)
\end{pmatrix}
\]  \hspace{1cm} (4.15)

Substituting Eq.(4.15) in Eq.(4.12) and collecting terms, we may end with:

\[
\begin{pmatrix}
1 - \gamma_1 \frac{1}{\gamma_2} \tan(\gamma_2 h_2) \cdot \tan(\gamma_1 h_1) + \gamma_0 \frac{1}{\gamma_1} \tan(\gamma_1 h_1) + \gamma_0 \frac{1}{\gamma_2} \tan(\gamma_2 h_2) - 1 \\
-\gamma_2 \tan(\gamma_2 h_2) - \gamma_1 \tan(\gamma_1 h_1) + \gamma_0 - \gamma_0 \frac{1}{\gamma_1} \gamma_2 \tan(\gamma_2 h_2) \cdot \tan(\gamma_1 h_1) \gamma_1
\end{pmatrix} \begin{pmatrix} A \\ F \end{pmatrix} = 0
\]  \hspace{1cm} (4.16)

To be satisfied, Eq.(4.16) requires that the determinant of the left matrix must vanish. This gives:
\[
\frac{1}{\gamma_1 \gamma_2} \left( \gamma_0 \gamma_2^2 \tan(\gamma_1 h_1) \tan(\gamma_1 h_1) + \tan(\gamma_1 h_1) \tan(\gamma_2 h_2) \gamma_1^2 \gamma_3 - \tan(\gamma_2 h_2) \gamma_0 \gamma_1 \gamma_3 \right) = 0
\] (4.17)

Solving Eq.(4.17) for \( \tan(\gamma_1 h_1) \):

\[
\tan(\gamma_1 h_1) = \frac{\gamma_1 \left( \tan(\gamma_2 h_2) \gamma_0 \gamma_3 - \tan(\gamma_2 h_2) \gamma_2^2 + \gamma_0 \gamma_2 + \gamma_3 \gamma_2 \right)}{\tan(\gamma_2 h_2) \gamma_0 \gamma_3 + \tan(\gamma_2 h_2) \gamma_1^2 \gamma_3 - \gamma_0 \gamma_2 \gamma_3 + \gamma_1^2 \gamma_2}
\] (4.18)

Following some mathematics and rearranging, one gets:

\[
\tan(\gamma_1 h_1) = \frac{\gamma_1 \left( (\gamma_3 \tan(\gamma_2 h_2)) + \gamma_2 \right) \gamma_0 - \tan(\gamma_2 h_2) \gamma_2^2 + \gamma_3 \gamma_2}{(\tan(\gamma_2 h_2) \gamma_2^2 - \gamma_2 \gamma_3) \gamma_0 + \tan(\gamma_2 h_2) \gamma_1^2 \gamma_3 + \gamma_1^2 \gamma_2}
\] (4.19)

Now dividing all terms in the right side \((\gamma_3 \tan(\gamma_2 h_2) + \gamma_2)\):

\[
\tan(\gamma_1 h_1) = \frac{\gamma_1 \left( \gamma_0 + \frac{-\tan(\gamma_2 h_2) \gamma_2^2 + \gamma_3 \gamma_2}{\gamma_3 \tan(\gamma_2 h_2) + \gamma_2} \right)}{(\tan(\gamma_2 h_2) \gamma_2^2 - \gamma_2 \gamma_3) \gamma_0 \gamma_2 + \tan(\gamma_2 h_2) \gamma_1^2 \gamma_3 \gamma_2 + \gamma_1^2 \gamma_2}
\] (4.20)

Dividing the denominator of Eq.(4.20) by \( \gamma_2 \) and then simplification the right term and rearrange, we may write the denominator as:

\[
\frac{-\gamma_3 \gamma_0}{\gamma_3 \gamma_2 + \gamma_2} + \frac{-\tan(\gamma_2 h_2) \gamma_0 \gamma_2}{\gamma_3 \gamma_2 + \gamma_2}
\] (4.21)

Dividing all term by \( \gamma_2^2 \) and substituting the last in Eq.(4.20), then:
\[
\tan (\gamma_1 h_1) = \frac{\gamma_0 + \frac{\gamma_3}{\gamma_1} \left( \frac{\gamma_3 \tan (\gamma_2 h_2)}{\gamma_2} + 1 \right) - \frac{\gamma_2 \tan (\gamma_2 h_2)}{\gamma_1} \left( \frac{\gamma_3 \tan (\gamma_2 h_2)}{\gamma_2} + 1 \right)}{1 - \frac{\gamma_0 + \frac{\gamma_3}{\gamma_1} \left( \frac{\gamma_3 \tan (\gamma_2 h_2)}{\gamma_2} + 1 \right) - \frac{\gamma_2 \tan (\gamma_2 h_2)}{\gamma_1} \left( \frac{\gamma_3 \tan (\gamma_2 h_2)}{\gamma_2} + 1 \right)}{\gamma_1}}
\]  
(4.22)

Making use of the relation: \( \tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \) we can rearrange Eq.(4.22) as:

\[
\gamma_1 h_1 = \tan^{-1} \frac{\gamma_0}{\gamma_1} + \tan^{-1} \left[ \frac{\gamma_2}{\gamma_1} \tan \left( \tan^{-1} \frac{\gamma_3}{\gamma_2} - \gamma_2 h_2 \right) \right] + m \pi, m = 0, 1, 2, ...
\]  
(4.23)

Defining \( Q_2 \) such that: \( Q_2 = \frac{\gamma_3}{\gamma_2} - \gamma_2 h_2 \) then:

\[
\gamma_1 h_1 = \tan^{-1} \frac{\gamma_0}{\gamma_1} + \tan^{-1} \left[ \frac{\gamma_2}{\gamma_1} \tan Q_2 \right] + m \pi
\]  
(4.24)

Which is simplified to the normal well-known dispersion relation:

\[
\gamma_1 h_1 = \tan^{-1} \left( \frac{\gamma_0}{\gamma_1} \right) - \tan^{-1} \left( \frac{Q_2}{\gamma_1} \right)
\]  
(4.25)

The term \( \tan^{-1} \left( \frac{\gamma_2}{\gamma_1} \tan Q_2 \right) \) can be rewritten as:

\[
\tan^{-1} \left( \frac{\gamma_2}{\gamma_1} \tan Q_2 \right) = \tan^{-1} \left[ \left( 1 - \frac{\gamma_1 - \gamma_2}{\gamma_1} \right) \tan Q_2 \right]
\]  
(4.26)
Since the term $\frac{\gamma_1 - \gamma_2}{\gamma_1}$ in Eq.(4.26) is in general much less than unity, then using Taylor expansion

$$\tan^{-1}\left( (1-a) \tan x \right) \approx x - a \left( x + \frac{2x^3}{3} - \frac{2x^3}{15} \right) \approx x - \frac{a}{2} \sin 2x, a \ll 1 \quad (4.27)$$

and performing some mathematical manipulations, one may have [30]

$$\tan^{-1}\left( \frac{\gamma_2}{\gamma_1} \tan Q_2 \right) \approx Q_2 - \frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2} \sin 2Q_2 = Q_2 - r \sin 2Q_2 = Q_2 - Q_r \quad (4.28)$$

Such $r$ is represents the Fresnel's reflection coefficient [56] of the light at the first interface and $Q_r = r \sin 2Q_2$. Substituting from Eq.(4.28) in Eq.(4.23) above then the final dispersion equation becomes:

$$\gamma_1 h_1 + \gamma_2 h_2 = \tan^{-1} \frac{\gamma_0}{\gamma_1} + \tan^{-1} \frac{\gamma_2}{\gamma_1} - Q_r, m = 0, 1, 2, ... \quad (4.29)$$

Indeed when the two inner media are identical we would have:

$$\gamma_1 (h_1 + h_2) = \gamma_1 h = \tan^{-1} \frac{\gamma_0}{\gamma_1} + \tan^{-1} \frac{\gamma_2}{\gamma_1} + m \pi \quad (4.30)$$

Which is the same relation of the three layer structure.

### 4.3 Power Flow

The time-average power flowing in asymmetric four-layer slab waveguide is given from Eq.(2.41) so in chapter 2 as:

$$P_{av} = \frac{1}{2\omega} \int_{-\infty}^{\infty} \Re \left[ \frac{\beta}{\mu} |E|^2 \right] dx \quad (4.31)$$
Making use of Eq.(4.2) above then the individual power flowing in each layer is given as:

\[
P_0 = \frac{1}{4} \frac{\beta A^2 \cos(a)}{\omega \mu q_0}
\]

\[
P_1 = \frac{1}{4} \frac{\gamma_1}{\omega \mu_1}
\]

\[
P_2 = \frac{1}{4} \frac{\beta AB^2 (h_1 + \sin(\gamma h_1) \cos(-\gamma h_1 + 2q))}{\omega \mu_2}
\]

\[
P_3 = \frac{1}{4} \frac{\beta AB^2 \cos(a)}{\omega \mu q_3}
\]
4.4 Result and Discussion

The structure under consideration is an asymmetric four layer slab waveguide with two LHM layers each having negative dielectric permittivity $\varepsilon_1$ and $\varepsilon_3$, and negative magnetic permeability $\mu_1$ and $\mu_3$ in the applicable frequency range. The electric permittivity and magnetic permeability of LHM are given by [54, 57, 58]:

\[
\varepsilon(\omega) = \varepsilon_0 \left( 1 + \frac{1}{\varepsilon_0 N \alpha^{-1} \left[ \varepsilon_1 + i \left( \frac{k_b^2}{\alpha \varepsilon_2} \right) \right] - 1/3} \right)
\]  \hspace{1cm} (4.36)

\[
\mu(\omega) = \mu_0 \left( 1 + \frac{1}{N \alpha^{-1}_{mm} + i (k_b^2) / \alpha \varepsilon_2 - 1/3} \right)
\]  \hspace{1cm} (4.37)

where

\[
\alpha_{mm}^{-1} = \frac{4 \varepsilon_b}{N k_b^2 R^2} \alpha - i \left( \frac{k_b^3}{6\pi} - \frac{2k_b}{3\pi NR^2} \right) + \frac{1}{16\pi N k_b^3 R^5} \sum_{l \neq n}^N 3 + \cos[2\pi(l-n)/N] \sin[\pi(l-n)/N]^{3}
\]  \hspace{1cm} (4.38)
Where $N_d$ is number density of the loop inclusions per unit volume, $\alpha_{ee}$ is electric polarizability of the loop, and $k_0 = \omega \sqrt{\varepsilon_0 \mu_0}$ is the background wave number and $\alpha_{mm}$ is the magnetic polarizability of the nano-ring, $N$ number of the identical nano-spheres with radius $a$, $\lambda_b$ is the wave length in the background medium, $R$ is the circle’s radius, $\varepsilon_b$ back ground material dielectric constant, $k_b = \omega \sqrt{\varepsilon_b \mu_0}$ is the background wave number, $n = 1, 2, ... N$, and $\alpha$ is given by,

$$\alpha = \left[ \left(4\pi\varepsilon_b a^3 \frac{\varepsilon - \varepsilon_b}{\varepsilon + 2\varepsilon_b}\right)^{-1} - i \frac{k_b^3}{6\pi\varepsilon_b} \right]^{-1}$$

The material exhibited both negative effective permeability and negative permittivity, in the frequency range 6.3 THz to 6.34 THz, at optical frequency for the following parameters: $R = 40nm, a = 16nm, N = 6, N_d = (108nm)^{-3}$, and $\varepsilon_b = 2.2\varepsilon_0$. These values would give rise to both effective permittivity and permeability have negative real parts simultaneously at 6.3-6.34 terahertz. Using Maple, a computer code has been developed to solve the dispersion relation numerically and compute the effective refractive index so that the dispersion curves could be plotted. It has been also designed to plot the field profiles for various configurations and the power flow within the structure against different parameters. In the discussion below, the LHM medium is selected according to the analysis of a previously published research paper [54]. In some discussion, however, the LHM medium is chosen depending on the previous values. Permeability and Permittivity for the other layers are generally given as: $\varepsilon_0 = 2.17$, $\varepsilon_2 = 2.17$ and the media are considered nonmagnetic so that $\mu_0 = \mu_2 = 1$. 
The material exhibited both negative effective permeability and negative permittivity, in the frequency range 6.32 to 6.34 terahertz. The dielectric constant for $BaF_2$ can be obtained from Sellmeier dispersion relationship, which is given by [54]:

\[
\varepsilon(\lambda) = 1 + \frac{A_1 \lambda^2}{\lambda^2 - \lambda_1^2} + \frac{A_2 \lambda^2}{\lambda^2 - \lambda_2^2} + \frac{A_3 \lambda^2}{\lambda^2 - \lambda_3^2} + \ldots
\] (4.40)

where $A_1$, $A_2$, $A_3$, $\lambda_1$, $\lambda_2$ and $\lambda_3$ are called Sellmeier coefficients, and have the following values for $BaF_2$ glass: $A_1 = 0.63356$, $A_2 = 0.506762$, $A_3 = 3.8261$, $\lambda_1 = 0.057789$, $\lambda_2 = 0.109681$, and $\lambda_3 = 46.38642$. It can be seen from Fig.(4.3) that the real refractive index decreases with increasing frequency. This means that the group velocity is negative, and the behavior is LHM. The effective refractive index is shown to be higher valued in lower modes. Accordingly, lots of applications can be established: anti-reflective coating and photonic devices such as light emitting diodes (LEDs) and image sensors are some examples. There are no solutions when the
frequency is less than $6.32 \times 10^{14}$ Hertz and greater than $6.34 \times 10^{14}$ Hertz. In addition, increasing the frequency, the real part of the effective index gradually decreases. That is, the structure becomes much sensitive to the frequency change and the confinement to the guide decreases.

![Fig(4.4): Dispersion curves: Im(N) against frequency for: a) TE$_0$ (solid line), b) TE$_1$ (dashed) and c) TE$_2$ (dotted), Thickness h$_1$=800nm, h$_2$=200nm.](image)

It is worth to determine the exponential decay of the wave along z-direction. The imaginary part of the effective refractive index is plotted against frequency for different modes in Fig.(4.4). This is expected to shed light on the extinction or absorption coefficient. Three modes, namely $TE_0$, $TE_1$ and $TE_2$ are considered for the same parameters adopted in Fig.(4.3) above. It is clear that the imaginary part of the refractive index increases as frequency increases. That is, absorption increases with frequency which is reasonable. A maximum value of the refractive index is achieved at about 6.326 terahertz for the three modes. This is the frequency at which the
extinction coefficient has its highest achievable value. Thus, operating the guide at other frequencies than this value would be more efficient. It is clear from the Fig(4.4) that the absorption is higher for lower modes of the structure. Consequently, higher modes are not suitable for designing of long distance wave guiding devices.

![Diagram](image.png)

**Fig.(4.5):** Modal field distribution of the structure under last consideration for TE

The plot shows the modal field distribution of the proposed waveguide in Fig.(4.5), for the parameters given in Fig.(4.3). The field distributions are in expected shape for the fundamental mode \((m = 0)\). Figure shows that the electric field is confined in the second and third layers. Besides that, we have underdamped (weak) radiating modes, these modes rapidly diminish in the cladding and substrate layers.
We explore the effect of the mode order on the total power in the structure. In Fig.(4.6), we plot the total power for different modes versus the allowed frequency range for thickness $h_1 = 800$ nm and $h_2 = 400$ nm. The total power is higher for large order of mode and gets lower at fundamental mode which means that the greatest amount of energy will be carried in the highest order.
In Fig.(4.7), the ratio of power flow within cladding and substrate are shown for the same previous parameters given in Fig.(4.3). As seen from Fig.(4.7.a), the power in cladding layer decreases with increasing the order of mode and the power decreases as the frequency increases. Fig.(4.7.b) also shows that the power in the substrate decreases with increasing frequency for $m = 0.1$ but for $m = 2$ the power increases with increasing the frequency.
In Fig. (4.8), we study the penetration depth of the structure to measure how deep electromagnetic radiation can penetrate into the material. The relation between the penetration depth ($\Delta x$) and frequency for different values of mode is studied. From Fig. (4.8), it can be seen that the penetration depth decreases gradually with increasing frequency, which has the largest penetration depth at lower frequency.
In this case, the dispersion properties of the proposed structure are studied when the first and second layers are LHM and the other layers are RHM with positive
parameters. We consider a left handed materials with $\varepsilon_1 = -6.22 + 0.04i$, $\mu_1 = -1 + 0.002i$, $\varepsilon_2 = -11.18 + 0.06i$, $\mu_2 = -1 + 0.002i$. Dielectric permittivity and magnetic permeability for other layers are generally given as: $\varepsilon_0 = 3.21 + 0.02i$ and $\varepsilon_3 = 4.14 + 0.03i$ and the media are considered nonmagnetic so that $\mu_0 = \mu_3 = 1$.

It is significant to study the effective refractive index with different guiding layer thickness. From Fig (4.10), it can be seen that the real refractive index increases with increasing frequency at $TE_0$ mode. This means that the group velocity is positive. The effective refractive index is shown to be higher valued for higher thickness. Fig(4.10) illustrates the change of the real part refractive index is very small and increases with increasing the thickness. This means that group velocity is positive, and the behavior of the structure is RHM. Based on previous values, this combination supports only one mode i.e. $TE_0$ mode, and does not support other higher mode. Thus, all energy is carried by the fundamental mode.
To study the absorption properties of the structure, the relation between the imaginary part of the modal effective index and frequency is carried out in Fig.(4.11).

It is clear that the imaginary part of the refractive index decreases as frequency increases. That is, absorption decreases with frequency and sensitivity of the structure is expected to increase.
In Fig. (4.12), the modal field distribution of the proposed waveguide for the same previous parameters given in Fig.(4.10) is shown. As seen from figure, the electromagnetic field confined to the second and third layers. Fig.(4.12) also shows that the tails represent the distribution of an evanescent electric field in substrate and cladding.
In Fig.(4.13), we plot the total power in structure for $TE_0$ mode versus the allowed frequency range for different core’s radii. This shows that the total power has the highest value at lower frequencies. Fig.(4.13) also shows that the total power flow in the structure decreases as the thickness increasing.
In Fig.(4.14.a), we plot the fraction of power flow in cladding for $TE_0$ mode versus the allowed frequency range for different core radii. This shows that the fraction of power flow decreases as the frequency increasing. From Fig.(4.14.b), it can be seen the fraction of power in substrate increases as increasing the frequency. It is clear that from Fig.(4.14.a) and Fig.(4.14.b) that power in cladding and substrate increases as the thickness increases.
Fig(4.15) : Penetration depth ($\Delta x$) for different thicknesses: a) $h_1=500\text{nm}$ (solid line), b) $h_1=550\text{nm}$ (dashed) and c) $h_1=600\text{nm}$ (dotted). Thickness $h_2=100\text{nm}$, $\varepsilon_0=3.21+0.02i$, $\varepsilon_1=-6.22+0.04i$, $\mu_1=1+0.002i$, $\varepsilon_2=-11.18+0.06i$, $\mu_2=1+0.002i$, $\varepsilon_3=4.14+0.03i$.

To see how deep the electromagnetic radiation can travel into the material, the penetration depth ($\Delta x$) is plotted against frequency for different values of thickness. This is illustrated in Fig.(4.15). It can be seen that the penetration depth gets larger with increasing both frequency and thickness.
CHAPTER 5

$TM$ Modes in Asymmetric Multi-Layered Planar Waveguide
5.1 Introduction

In this chapter, the propagation of a transverse magnetic plane polarized wave (TM) incidence on two different waveguide structures with a left-handed material is studied. The effect of frequency and thickness on the modal effective refractive index of the structure is also studied. In addition, the electric field profile, and the power distribution in each layer are shown.

5.2 Theory

The homogenous wave equation for $\vec{H}$ is

$$\nabla^2 \vec{H} + k^2 \vec{H} = 0 \quad (5.1)$$

where the wavenumber $k$ is defined by $k^2 = \omega^2 \varepsilon \mu$.

Solutions of Eq. (5.1) for $H_y$ in the four layers waveguide structure are given by:

$$H_y = \begin{cases} 
A \exp(q_0 x) & ; -\infty < x \leq 0 \\
B \exp(i q_1 x) + C \exp(-i q_1 x) & ; 0 \leq x \leq h_1 \\
D \exp[i q_2(x - h_1)] + E \exp[-i q_2(x - h_1)] & ; h_1 \leq x \leq h_2 \\
F \exp(-q_3(x - h_1 - h_2)) & ; h_1 + h_2 \leq x < \infty 
\end{cases} \quad (5.2)$$

We can write the magnetic field equations as:

$$H_0 = A_0 \exp(q_0 x)$$
$$H_1 = A_1 \cos(q_1 x - a_1)$$
$$H_2 = A_2 \cos(q_2(x - h_1) + a_2)$$
$$H_3 = A_3 \exp(-q_3(x - h_1 - h_2)) \quad (5.3)$$

where $a_1$ and $a_2$ are phase differences which can be written as,
\[ a_i = \arctan \left( \frac{q_0}{q_1} \right) \]  
\[ a_2 = \arctan \left( \frac{q_3}{q_2} \right) \]  

where \( A_0, A_1, A_2 \) and \( A_3 \) are the wave amplitudes in the four layers that can be determined from the boundary conditions. The parameters \( q_0, q_1, q_2 \) and \( q_3 \) are given by:

\[ q_0^2 = k_0^2 \left( N^2 - \varepsilon_0 \mu_0 \right) \]
\[ q_1^2 = k_0^2 \left( \varepsilon_1 \mu_1 - N^2 \right) \]
\[ q_2^2 = k_0^2 \left( \varepsilon_2 \mu_2 - N^2 \right) \]  

and

\[ q_3^2 = k_0^2 \left( N^2 - \varepsilon_3 \mu_3 \right) \]  

Applying Eq.(2.60), we can have [56-59]:

\[
\begin{pmatrix}
  F \\
  -q_3 F
\end{pmatrix} = M_1 M_2 \begin{pmatrix}
  A \\
  q_0 A
\end{pmatrix}
\]  

with:

\[
M_1 = \begin{pmatrix}
  \cos(q_1 h_1) & \frac{\varepsilon_1}{q_1} \sin(q_1 h_1) \\
  -\frac{q_1}{\varepsilon_1} \sin(q_1 h_1) & \cos(q_1 h_1)
\end{pmatrix}
\]  

and
The structure transfer matrix (i.e. the product $M_1M_2$):

$$M_2 = \begin{pmatrix}
\cos(q_2h_z) & \frac{\varepsilon_2}{q_2} \sin(q_2h_z) \\
-q_2 \frac{\varepsilon_2}{\varepsilon_2} \sin(q_2h_z) & \cos(q_2h_z)
\end{pmatrix}$$  \hfill (5.9)

Substituting Eq.(5.10) in Eq.(5.7) and collecting terms, then solving $\tan(q_1h_1)$, and following the steps outlined in chapter 2, we will end up with the dispersion relation for $TM$ mode:

$$q_1(h_1 + h_2) = \tan^{-1} \frac{\varepsilon_0q_0}{\varepsilon_0q_1} + \tan^{-1} \frac{\varepsilon_2q_2}{\varepsilon_2q_1} + m\pi$$  \hfill (5.11)

As a special case, when the two inner media are identical we would have the dispersion relation for three layers as mentioned in chapter 3:

$$q_1(h_1 + h_2) = q_1h = \tan^{-1} \frac{\varepsilon_0q_0}{\varepsilon_0q_1} + \tan^{-1} \frac{\varepsilon_2q_2}{\varepsilon_2q_1} + m\pi$$  \hfill (5.12)

For sensing applications, the most important parameter for optical waveguide sensor is the fraction of total power flowing in the upper layer. The time-averaged power flowing in the four layers of the waveguide is given by Eq. (2.40). Using of Eq. (5.3), we could find the power flow in each layer, that is [60-61]:

$$P_0 = \frac{1}{4} \frac{\beta A^2 \cos(a_i)^2}{\omega \mu_0 q_0}$$  \hfill (5.13)
\[ P_1 = \frac{1}{4} \frac{\beta A^2 (h_1 + \sin(q_1 h_1) \cos(-q_1 h_1 + 2a_1))}{\omega \mu_1} \]  
(5.14)

\[ P_2 = \frac{1}{4} \frac{\beta A B^2 (h_2 + \sin(q_2 h_2) \cos(-q_2 h_2 + 2a_2))}{\omega \mu_2} \]  
(5.15)

\[ P_3 = \frac{1}{4} \frac{\beta A B^2 \cos(a_2)^2}{\omega \mu_3 q_3} \]  
(5.16)

Again, applying the power expression given by Eq.(2-36) to the solutions of Helmholtz equation in the three layers given by Eqs.(3-1) to (3-3). Carrying out this integration, one can finally have:

\[ P_0 = \frac{1}{2\omega} \int_{-\infty}^{0} \text{Re} \left[ \frac{\beta}{\mu} |A e^{q_0 x}|^2 \right] dx = \frac{1}{4} \frac{\beta A^2}{\omega \mu_0 q_0} \]  
(5.17)

\[ P_1 = \frac{1}{2\omega} \int_{0}^{\infty} \text{Re} \left[ \frac{\beta}{\mu} B \cos(q_1 x) \right] dx = \frac{1}{4} \frac{\beta B^2 (w + \sin(q_1 w) \cos(-q_1 w + 2a_1))}{\omega \mu_1} \]  
(5.18)

\[ P_2 = \frac{1}{2\omega} \int_{0}^{\infty} \text{Re} \left[ \frac{\beta}{\mu} |C e^{(-q_2 x - w)}|^2 \right] dx = \frac{1}{4} \frac{\beta C^2}{\omega \mu_2 q_2} \]
5.3 Result and Discussion for Three Layer Slab (TM) Mode

![Schematic diagram of a three-layers slab waveguide structure including ENG substrate](image)

In this case, the dispersion properties of the proposed structure are studied when the core and cladding are RHM and the substrate is epsilon negative material with negative parameters for TM mode. We consider a ENG substrate with \( \varepsilon_2 = -2.5 + 0.39i, \mu_2 = 1 - 0.004i \). Dielectric permittivity and magnetic permeability for the core and substrate are generally given as: \( \varepsilon_0 = 2.99, \varepsilon_1 = 3.38 \) and the media are considered nonmagnetic so that \( \mu_0 = \mu_1 = 1 \).
It is significant to study the propagation of waves in a three-layer slab waveguide containing ENG materials in substrate. Fig.(5.2) shows the relation between the real refractive index and the operating frequency from 3.4 to 3.8 terahertz, for different thicknesses. We found that there is propagation (numerical solution) in a narrow frequency range, between 3.4 to 3.8 terahertz, and there is no propagation (no solution) for higher modes order. We notice that, real refractive index for $TM_1$ is higher than from $TE_1$. 

\[ \varepsilon_1 = 3.38, \omega_p = 1.2 \times 10^{16}, \omega_0 = 1.2 \times 10^{14}, F = 0.56. \]
Fig.(5.3) : Dispersion curves: Imaginary refractive index against frequency thicknesses : a) $TM_1$ (solid line), b) $TE_1$ (dashed), $w=[900-950]nm, \varepsilon_0 = 2.99, \mu_2 = 1 - 0.004I, \varepsilon_1 = 3.38, \omega_p = 1.2 \times 10^{16}, \omega_0 = 1.2 \times 10^{14}, F = 0.56.$

It is clear that attenuation (the imaginary part of the refractive index) increases as frequency increases, which means higher absorption at higher frequencies. The change of the attenuation is very small and claims small value, which means we have low loss structure.
Fig. (5.4) Modal field distribution of the structure under consideration for TM$_1$, Thickness $w = 850\text{nm}$, $\varepsilon_0 = 2.99$, $\mu_2 = 1 - 0.004I$, $\varepsilon_1 = 3.38$, $\omega_p = 1.2 \times 10^{16}$, $\omega_0 = 1.2 \times 10^{14}$, $F = 0.56$.

The magnetic field profile for the fundamental mode ($m = 1$) is shown in Fig. (5.4) for the same previous parameters given in Fig. (5.2). Fig. (5.4) shows that the evanescent tails which represent the distribution of magnetic field in substrate and cladding. Moreover, this structure supports only one mode ($m = 1$) and does not support any other modes.
In Fig.(5.5.a), we plot the fraction of power flow in cladding for $TM_1$ mode versus the allowed frequency range for different core thickness. This shows that the fraction of power flow decreases as the frequency increases, and power decrease with core thickness as well. From Fig.(5.5.b), it can be seen the fraction of power in substrate decreases with frequency increase, and it is clear that from the power increases with core thickness increase.
5.4 Result and Discussion for Four Layer Slab (TM) Mode

We consider an asymmetric four layers slab waveguide with two LHM layers, in these material both dielectric permittivity $\varepsilon_1$ and $\varepsilon_3$, in addition to magnetic permeability $\mu_1$ and $\mu_3$ which gives negative results in the applicable frequency range. The electric permittivity and magnetic permeability of LHM are given by equations from Eq.(4.36 – 4.39). For TM modes, their dispersive equations are similar with that of the corresponding TE modes. But, their magnetic permeability in the equations is replaced by dielectric permittivity [43].

Fig.(5.6): Schematic diagram of Four-layers with a left-handed metamaterial
Fig (5.7) illustrates the dependence of the real part of the effective index for both TM$_2$, TE$_2$ modes in a narrow frequency band. For simplicity, we assume that waveguide thickness $h_2$ are fixed at 400 nm but the thickness $h_1$ is 800 nm. Through this figure, it is clear that, the real part of the effective index is decreasing as the frequency increases. The real part of the effective index exhibits the highest achieved value in the case of TM$_2$. 
Fig(5.8) : Dispersion curves: Im(N) against frequency for: a) TM$_2$ (solid line), b) TE$_2$ (dashed), Thickness $h_1=800\text{nm}$, $h_2=400\text{nm}$.

It is worth to determine the exponential decay of the wave along z-direction. Dispersion equation (Eq.5.11) is numerically solved for the second mode, and the relation between the imaginary part of the modal effective index and frequency is plotted in Fig.(5.8). It is clear that the imaginary part of the refractive index increases as frequency increases. That is, absorption increases with frequency which is reasonable. It is clear from the figure that the absorption is higher for TM$_2$ than TE$_2$. of the structure.
Fig.(5.9) Modal field distribution of the structure under last consideration for TM$_2$

Fig.(5.9) shows the modal field distribution of the proposed waveguide in Fig.(5.9). For the same previous parameters given in Fig.(5.7) is used. The field distributions are in expected shape for the fundamental mode($m = 2$). As seen from figure, the electromagnetic field confined to the second and third layers. Fig.(5.9) also shows that the underdamped radiating mode of magnetic field in substrate and cladding.
In Fig.(5.10), the ratio of power flow within clad and substrate are shown for the parameters given in Fig.(5.7). As seen from Fig.(5.10.a), the power in cladding layer decreases with increasing the order of mode and the power decreases as the frequency increases. Fig.(5.10.b) also shows that the power in the substrate increases with increasing frequency for $m = 2, 3$ but for $m = 1$ the power decreases with increasing the frequency.

Fig(5.10): Power flow: a) TM$_1$ (solid line), b) TM$_2$ (dashed) and c) TM$_3$ (dotted), Thickness $h_1=800\text{nm}, \ h_2=400\text{nm}$. 
CONCLUSION

Asymmetric three-layer and four-layer slab waveguide structures have been studied for TE and TM modes. We extensively studied the following characteristics: effective refractive index, attenuation, electric field profile, power flow, confinement, and penetration depth for different mode order and different thicknesses in terahertz region of spectrum. We had many observations on a three-layer and four-layer slab waveguides. We list them in the following.

Remarks observed on asymmetric three-layer waveguides:

1. In the first structure, the guiding layer was assumed to be left handed Material. The results showed that the effective refractive index is high and exhibits higher values for higher modes, the attenuation factor is considerably high, but the power confinement factor is fascinating. Thus, the possible use of this structure in all optical circuits as retarder (to cause certain time delay for synchronization purposes). Therefore, this structure is good for transmitting signals for short distances.

2. In the second structure, the substrate was ENG material. The results showed that the low effective refractive index, low extinction coefficient (the change of the imaginary part of refractive index is very small) and decreases as thickness increases, and low power confinement. Thus, the structure is a weak candidate for wave guiding for a long distance.

3. In the third structure, the cladding was ENG material. The results showed low effective refractive index, low power attenuation, and high power confinement factors. Besides that, this structure supports only the fundamental mode \((m = 0)\) and does not support other modes, thus all the incident power can be carried in this mode. The structure is good for long distance wave guiding.

Remarks observed on asymmetric four-layer waveguides:

1. The second and fourth layers structure are left handed material. The material exhibited both negative effective permeability and negative permittivity, in the
frequency range 6.30 THz to 6.34 THz, The result showed low effective refractive
index, low absorption but it gets higher for higher modes. Consequently, higher
modes are not suitable for designing of long distance devices. From result we can see
the power is confined in second and third layers and the greatest amount of energy
will be carried in the highest order.

2. The second and third layers was left handed material. In this case, we assume the
permeability and permittivity are constant. The result showed that low effective
refractive index, the absorption is very small. Consequently, this structure is suitable
for designing of long distance optical devices. From result we can see the power is
confined in second and third layers and absorption decreases with frequency and
sensitivity of the structure is expected to increase.

For the case of $TM$ mode, the following remarks have been observed:
1. The results showed that the real part of the effective refractive index increases
with increasing frequency and that the imaginary part increases as the thickness of
the guided layer increases. The results show that we have low loss structure.

2. The power fraction flowing in the cladding versus the thickness of the guided layer
decreases with increasing the guided layer thickness and the frequency.

3. The power fraction flowing in the substrate layer decreases with increasing the
guided layer thickness.
REFERENCES


[16] Lapine, M., (2009), "Tunable Metamaterials: the Key Step to Practical Application".


Slab Waveguide Structure", The Islamic University Journal (Series of Natural Studies and Engineering), Vol.19, no.1, PP. 57-70.

[38] Isaac, T., (2009), "Tunable Plasmonic Structures for Terahertz Frequencies", a thesis for the degree of Doctor of Philosophy in Physics.


