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Fractional Universe Model Free of Cosmological Problems

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Abstract*: In this paper, we propose a new cosmological model from fractional action integral approach. Einstein's equations are derived from the fractional variational problem using Robertson-Walker metric to construct a model free from cosmological problems. In this approach, we have solved cosmological constant, entropy, flatness and horizon problems. We found that our model gives a better solutions than other model.*

Keywords*: Fractional universe, Cosmological problems.*

Introduction:

The cosmological standard Big Bang (SBB) model based on the general relativity (GR) has successfully explained the most important astronomical observations. Despite these successes, the SBB model has certain short comings. These include: Cosmological constat, entropy, Horizon flatness and age problems [1]. Many attempts had been made by several authors [2-10] to overcome these problems. Most of these attempts did not leave the framework of Einstein's model, and they were made either by modifying the theory of GR, or by introducing an additional term to its solutions. This extra term representing the vacuum energy-momentum tensor and is expressed as a cosmological constant[11].

It is desirable to have a classical model to solve these problems. The starting point in our model is to consider the a fractional space-time dimensions using the concepts of the scaling method introduced by Mandelbrot [12]. The variational principle is used in reference [13] to obtain the modified Euler-Lagrange equation of motion in fractional space -time coordinated by introducing the the fractionality in time as a scale parameter [12] $0<\alpha\le 1$.

In this paper, we use the Mandelbrot concepts [12] of scaling the time to propose a new cosmological model in order to solve all the cosmological problems which are mentioned previously. This paper is organized as follow: In section 2, a new cosmological constant is proposed. In section 3 the entropy problem is investigated by the three curvature parameters $k[0,\pm 1]$. In section 4 the flatness is investigated by the curvature parameters $k[0,\pm 1]$ and the universe homogeneity. Section 5 is dedicated to our conclusions.

2 Scale factor dependent cosmological constant

Let us first introduce a brief derivations to the fractional equation of Friedmann model (FRW), and then we will study the cosmological constant when $k=[0,\pm 1]$, this will be used in the next section to investigate the entropy problem.

Let us consider the variation of the action integral [13]

$$S^{\alpha}(q)=\int_a^b \mathrm{L}(q_i(\tau),\dot{q}_i(\tau),\tau)d^{\alpha}\tau \qquad i=1,2,3,....,n, \tag{1}$$

where $a<\tau<b$, and $0<\alpha\le 1$. Here we note that when $\alpha\to 1$, the functional $S^{\alpha}(q(t))$ is just the well known classical action function in classical mechanics. The "action" of variational principle can be chosen to be the invariant, requiring that $\delta S^{\alpha}=0$. Finally we obtained after these considerations the modified equations of motion as [13]

$$\frac{\partial \mathrm{L}}{\partial q_i}-\frac{d}{d\tau}\left(\frac{\partial \mathrm{L}}{\partial \dot{q}_i}\right)-\frac{\alpha-1}{\tau}\left(\frac{\partial \mathrm{L}}{\partial \dot{q}_i}\right)=0. \tag{2}$$

One should notice that the time-dependent damping term $\frac{\alpha-1}{\tau}\left(\frac{\partial \mathrm{L}}{\partial \dot{q}_i}\right)$ is different from that obtained by EL-Nabulsi [14], in which our term do not contain any variable t and so it is valid for any time τ.

Let us consider the following Lagrangian from the Riemann geometry

$$\mathrm{L}=\frac{1}{2}g_{ij}\dot{q}_i\dot{q}_j, \tag{3}$$

where the metric g_{ij} is arbitrary function of coordinates and $\dot{q}_i = \frac{dq_i}{d\tau}$.

The equation of motion for this system results as

$$\ddot{q}_l + \frac{\alpha - 1}{\tau}\dot{q}_l + \Gamma^i_{ij}\dot{q}_i\dot{q}_j = 0, \qquad (4)$$

where $\Gamma^i_{ij} = \frac{1}{2} g^{lk}[\frac{\partial g_{ki}}{\partial q_i} + \frac{\partial g_{ik}}{\partial q_j} - \frac{\partial g_{ij}}{\partial q_k}]$, is the Christoffel symbol of second kind.

This equation can be put in the form

$$\partial_i \gamma^i + \frac{(\alpha - 1)}{\tau}\partial_i \upsilon^i + c^2 R_{00} = 0, \qquad (5)$$

where γ^i is the particle acceleration, υ_i is the particle velocity and $R_{ab} = g^{ij} R_{ajbi}$ is the Ricci tensor, with $R_{abcd} = \frac{1}{2}(g_{bc,ad} - g_{bd,ac} + g_{ad,bc} - g_{ac,bd}) + \Gamma^j_{ab}\Gamma_{bjd} - \Gamma_{bjd}\Gamma^j_{ac}$ is the covariant curvature tensor[15] and $g_{ij,k} = \frac{\partial g_{ij}}{\partial q^k}$. We make use of the Raychaudhuri expression scalar factor $\partial_i \upsilon^i = 3\dot{\theta} = 3\frac{\dot{a}}{a}$ where $a(t)$ is the cosmic scale factor which is unknown function in time. It was showed in [13] that equation (5) can be put in the form of poisson's equation

$$\partial_i \gamma^i = -4\pi G\rho, \qquad (6)$$

where ρ is the mass density in gravitational interaction, G is

$$R_{00} = 4\pi\rho G_{eff}/c^2, \qquad (7)$$

where G_{eff} is the effective Newton gravitational constant (ENGC) and is given by

$$G_{eff} = G(1 + \frac{3(1-\alpha)}{4\pi\rho G\tau}\frac{\dot{a}}{a}), \qquad (8)$$

then

$$\Delta G = \frac{3(1-\alpha)}{4\pi\rho G\tau}\frac{\dot{a}}{a}. \qquad (9)$$

In FRW cosmology, the term ΔG modifies the Friedmann equations 00-component in the presence of cosmological constant as follows [15]:

$$(\frac{\dot{a}}{a})^2 + \frac{2(\alpha-1)}{\tau}\frac{\dot{a}}{a} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, \quad (10)$$

now substituting $(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho_c$ and $\frac{\rho}{\rho_c} = \Omega$, we obtain the cosmological constant.

$$\Lambda = 3\frac{k}{a^2} - 3(\frac{\dot{a}}{a})^2[\Omega - 1] + \frac{6(\alpha-1)}{\tau}\frac{\dot{a}}{a}. \quad (11)$$

It is worthwhile to mention that according to the calculations [3,5,16-20] which follow the basis of general relativity,the cosmological constant depends only on a^{-2}. While in our calculations which based on the derivation of Einstien equation and the fractional calculus, we will have an additional term which depends on a^{-1}. In this new version we can consider a number of cases. It can be set to take the values of $k = [0\ ,\ \pm 1]$. According to these values one distinguishes between the cases of ***a Flat, Closed and Open*** universes , respectively. If $\alpha = 1$, $\rho = \rho_c$ we obtain the cosmological constant as in Özar-Taha model [3],

$$\Lambda = 3\frac{k}{a^2}. \quad (12)$$

Assume that the scale factor is

$$a(t) = a_0 + Dt \quad (13)$$

then $\dot{a} = D$,

where a_0 is the scale factor of the universe at $t = 0$ and D is a constant. Thus the universe is born with an initial value

$$\Lambda_0 = 3\frac{k}{a_0^2} - 3(\frac{\dot{a}}{a_0})^2[\Omega - 1] + \frac{6(\alpha-1)}{\tau}\frac{\dot{a}}{a_0}, \quad (14)$$

then using the value of the curvature parameters, and the density parameters $\Omega = [1, > 1, < 1]$ for ***a Flat, Closed and Open*** spaces respectively in the equation (11), we have the following three cases:

I. For a flat universe, $k = 0$ and $\Omega - 1 = 0$, we obtain

$$\Lambda = \frac{6D}{a}\frac{(\alpha-1)}{\tau}. \tag{15}$$

II. For a closed universe $k=1$ and $\Omega-1>0$, we obtain

$$\Lambda = \frac{3}{a^2} - 3(\frac{D}{a})^2[\Omega-1] + \frac{6D}{a}\frac{(\alpha-1)}{\tau}. \tag{16}$$

III. For an open universe , $k=-1$ and $\Omega-1<0$, we obtain

$$\Lambda = -\frac{3}{a^2} - 3(\frac{D}{a})^2[\Omega-1] + \frac{6D}{a}\frac{(\alpha-1)}{\tau}. \tag{17}$$

3 Entropy Problem

In view of equation (11), we investigate the entropy conditions for three curvature parameters $k[0,\pm1]$ and by using $\frac{dS}{dt} = -2\frac{\beta a^3}{T}\frac{d\Lambda}{dt}$ where S is the entropy, T is the temperature and β is constant.

$$\frac{d\Lambda}{dt} = \frac{-6k}{a^3}\dot{a} + (\frac{6D^2\dot{a}}{a^3})[\Omega-1] - (\frac{6D\dot{a}}{a^2})\frac{(\alpha-1)}{\tau} - \frac{6D}{a}\frac{(\alpha-1)}{\tau^2}, \tag{18}$$

$$\frac{dS}{dt} = \frac{12\beta}{T}[k\dot{a} - D^2\dot{a}(\Omega-1) + Da\dot{a}(\frac{\alpha-1}{\tau}) + Da^2(\frac{\alpha-1}{\tau^2})]. \tag{19}$$

I. For a flat universe, $k=0$ and $\rho=\rho_c$, $\Omega-1=0$, we have

$$dS = \frac{12D\beta}{T}[ada(\frac{\alpha-1}{\tau}) - a^2(\alpha-1)d(\frac{1}{\tau})] \geq 0, \tag{20}$$

for $\alpha \leq 1$, then the entropy, S, is increasing.

$$S = \frac{2D\beta}{T}[a^2(\frac{1-\alpha}{\tau})] + C. \tag{21}$$

II. For a closed universe, $k=1$ and $\rho>\rho_c$, and $\Omega-1>0$, we have

$$dS = \frac{12\beta}{T}[da - D^2da(\Omega-1) + Dada(\frac{\alpha-1}{\tau}) - Da^2(\alpha-1)d(\frac{1}{\tau})] \geq 0, \tag{22}$$

for $\alpha \leq 1$, and $D^2(1-\Omega)<1$ then the entropy, S, is increasing.

$$S = \frac{12\beta}{T}[a(1+D^2(1-\Omega)) + \frac{D}{2}a^2(\frac{1-\alpha}{\tau})] + C^{'}. \tag{23}$$

For large a, the second term is dominated.

III. For an open universe, $k=-1$ and $\rho<\rho_c$, and $\Omega-1<0$, we have,

$$dS=\frac{12\beta}{T}[-da-D^2da(\Omega-1)+Dada(\frac{\alpha-1}{\tau})-Da^2(\alpha-1)d(\frac{1}{\tau})]\geq 0, \quad (24)$$

for $\alpha\leq 1$, and $D^2(1-\Omega)>1$ then the entropy, S, is increasing.

$$S=\frac{12\beta}{T}[a(D^2(1-\Omega)-1)+\frac{D}{2}a^2(\frac{1-\alpha}{\tau})]+C''. \quad (25)$$

For large a, the second term is dominated.

4 Flatness Problem

In this section, we will discuss the Flatness problem. Using equation (10), we obtain

$$\rho-\rho_c=\frac{1}{8\pi G}[\frac{3k}{a^2}+\frac{6(\alpha-1)}{\tau}\frac{\dot{a}}{a}-\Lambda], \quad (26)$$

in view of this relation the matter density differs from the critical density for all values of curvature parameters, even if $\Lambda=0$.

I. For a flat universe, $k=0$, we have

$$\rho-\rho_c=\frac{1}{8\pi G}[\frac{6(\alpha-1)}{\tau}\frac{\dot{a}}{a}-\Lambda]\neq 0. \quad (27)$$

II. For a closed universe, $k=1$, we have

$$\rho-\rho_c=\frac{1}{8\pi G}[\frac{3}{a^2}+\frac{6(\alpha-1)}{\tau}\frac{\dot{a}}{a}-\Lambda]>0. \quad (28)$$

III.For an open universe, $k=-1$, we have

$$\rho-\rho_c=\frac{1}{8\pi G}[\frac{-3}{a^2}+\frac{6(\alpha-1)}{\tau}\frac{\dot{a}}{a}-\Lambda]<0. \quad (29)$$

If we set $\alpha=0$ and $\Lambda=0$, we come to the relation obtained by Friedmann model, and equation (26) can be written as

$$\rho-\rho_c=\frac{3}{8\pi G}[\frac{k}{a^2}+f(a)-\frac{\Lambda}{3}]. \quad (30)$$

From the above relation in our model there is no relation between matter density and critical density of the universe. This arises from the fractional term, and nongeometrical term (during the vacuum era).

The horizon problem solved by using Özer-Taha model [3] which gives the value of scale factor at $t = t_0 \to 0$ i.e. $a(0) \approx 10^{-20}$ cm, using this value in our model, which is very small compared to the current scale factor now $a(t) \approx 10^{28}$ cm, and this leads to conclude that the horizon radius $d_H(t)$ is greater than the radius of the universe $d(t)$, i.e. $(d_H(t) > d(t))$ which implies that the universe is homogeneous and isotropic from the early moments of the universe.

5 Conclusion

In this work we have proposed a modified cosmological model from fractional action integral approach with its main feature cosmological constants. It is worthwhile to notice that equation (11) gives a nonsingular behavior of the fractional equation for cosmological constant, which scale factor dependent for each value of curvature parameter k. This expression constitutes a generalization to other models. Steaming from this expression Λ we were able to give a reasonable solution for the entropy and the flatness problems. This means that the proposed model may advantageously and allow us to solve the controversies arising from the limitation of general relativity.

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