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A Subclass of P-valent Uniformly Convex Functions with negative coefficient Defined by a Certain Linear Operator

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Abstract*: In this paper a new subclass of p-valent uniformly convex functions with negative coefficients defined by a certain linear operator is introduced. Coefficient estimate, distortion theorems associated with fractional derivative operator are investigated for this class. Further class preserving integral operator, extreme points, radii of p-valently starlikely and convexity and other interesting properties for the said class have been determined.*

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Introduction:

Let S_p denote the class of functions of the form

(1.1) $$f(z)=z^p+\sum_{n=p+1}^{\infty}a_n z^n,$$

which are analytic and p-valent in the unit disk $U=\{z:|z|<1\}$. Also denote by T_p the class of functions the form

(1.2) $$f(z)=z^p-\sum_{n=p+1}^{\infty}a_n z^n \qquad (z\in U) \quad (a_n\geq 0),$$

which are analytic and p-valent in U.

A function $f(z)\in S_p$ is to be starlike of order $\alpha\,(0\leq\alpha<p)$, denoted by $S_p(\alpha)$, if and only if

(1.3) $$\operatorname{Re}\left\{\frac{z f'(z)}{f(z)}\right\} > \alpha \qquad (z \in U),$$

and it is called convex of order $\alpha (0 \le \alpha < p)$, denoted by $K(\alpha)$, if and only if

(1.4) $$\operatorname{Re}\left\{1+\frac{z f''(z)}{f'(z)}\right\} > \alpha \qquad (z \in U).$$

If $f(z)$ given by (1.1) and $g(z) \in S_p$ is defined by

(1.5) $$f(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n ,$$

then the convolution or Hadamard product of $f(z)$ and $g(z)$ is given by

(1.6) $$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} b_n a_n z^n .$$

A function $f(z) \in S_p$ is said to be $\boldsymbol{\beta}$ — uniformly starlike function of order α denoted by $\beta - S_p(\alpha)$ if and only if

(1.7) $$\operatorname{Re}\left\{\frac{z f'(z)}{f(z)} - \alpha\right\} \ge \beta \left|\frac{z f'(z)}{f(z)} - p\right|,$$

for some $\alpha(-1 \le \alpha < p)$ and all $z \in U$ and is said to be , $\boldsymbol{\beta}$ — uniformly convex of order α denoted by $\beta - K_p(\alpha)$ if and only if

(1.8) $$\operatorname{Re}\left\{1+\frac{z f''(z)}{f'(z)} - \alpha\right\} \ge \beta \left|\frac{z f''(z)}{f'(z)} - p\right| ,$$

for some $\alpha(-1 \le \alpha < p)$ and all ($z \in U$) .

The class $S_p(\alpha)$ *and* $K_p(\alpha)$ are introduced by Patil and Thakar [3], while the classes $S(\alpha)$ and $K(\alpha)$ were first studied by Reborston [8], Schild [1] , , and others .The classes $\beta - S_p(\alpha)$ and $\beta - K_p(\alpha)$ were introduced and studied by Goodman [2] , RØnning [5] , and others.

Let

(1.9) $$S^*{}_p(\alpha) = S_p(\alpha) \cap T_P , \; K_P^*(\alpha) = K_p(\alpha) \cap T_p ,$$

$$\beta - S^*{}_p(\alpha) = [\beta - S_p(\alpha)] \cap T_P, \text{ and}$$

$$\beta - K^*{}_p(\alpha) = [\beta - K_p(\alpha)] \cap T_P .$$

The classes $S*_p(\alpha)$ and $K^*{}_p(\alpha)$ have been studied by Silverman [6] and Silvia [7], and others.

The incomplete beta function $\phi_p(a,c;z)$ is defined by

(1.10) $$\phi_p(a,c;z)=\sum_{n=0}^{\infty}\frac{(a)_{n-p+1}\,z^{n+1}}{(c)_{n-p+1}},$$

For $c \neq 0,-1,-2,...,\ a \neq -1,-2,-3,...,$ where $(a)_n$ is the Pochhammer symbol defined by

(1.11)
$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}1\ , & n=0\\ a(a+1)(a+2)...(a+n-1)\ , & n\in N=1,2,...\end{cases}$$

The linear operator $L_p(a,c)$, on the class S_p is defined by

(1.12) $$L_p(a,c)\ f(z)=\ \phi_p(a,c;z)*f(z) =$$
$$z^p+\sum_{n=p+1}^{\infty}\frac{(a)_{n-p}}{(c)_{n-p}}a_n z^n\ ,\qquad z\in U\ .$$

The Linear operator $L(a,c)$ is defined by Carlson and Shaffer [4].

It may be noted that

$L_p(a,a)\ f(z)=f(z)$ and $L_p(2,1)\ f(z)=z\,f'(z)$, also

(1.13) $$L_p(m+1,m)\ f(z)=\ z^p+\sum_{n=p+1}^{\infty}\frac{(m+1)_{n-p}}{(1)_{n-p}}a_n\,z^n\ ,\qquad z\in U.$$

Definition 1. For $(-1\le\alpha<p)$ and $\beta\ge 0$, we let $S_p^n\ (\alpha,\beta)$ be the subclass of S_p consisting of function $f(z)$ of the form (1.1) and satisfying the following

(1.14) $$\left\{\frac{pL_p(m+1,1)(L_p(a,c)\,f(z))}{L_p(m,1)(L_p(a,c)\,f(z))}-\alpha\right\}>p\beta\left|\frac{L_p(m+1,1)(L_p(a,c)\,f(z))}{L_p(m,1)(L_p(a,c)\,f(z))}-1\right|,\ \ z\in U$$

Also let $T_p^n\ (\alpha,\beta)=S_p^n\ (\alpha,\beta)\ \cap T_p$.

It may be noted that the class $T_p^n\ (\alpha,\beta)$ extends the class of starlike, convex, prestarlike, β- uniformly starlike and β-uniformly convex by giving specific values of $\alpha,\beta,n,p,a,$ and c.

Here we mention the following important subclasses of the class $T_p^n\ (\alpha,\beta)$.

(i) For $a=c=m=p=1$ the class $T_p^n\ (\alpha,\beta)$ reduces to the class β - uniformly starlike functions.

(ii) For $a=2,\ c=m=p=1$, we obtain the class β - uniformly convex function.

(iii) The class of starlike function can be obtained by choosing $a=c=m=p=1$ and β =0, further the class of convex function can be obtained by choosing $a=2,\ c=m=p=1$, and β =0.

(iv) For $c=m=p=1$, and a=2- 2α we obtain the class pre-starlike function. Several other classes studied by various research workers can be obtained from the class $T_p^n\ (\alpha,\beta)$.

A class of fractional derivative operator

Following Raina and Nahar [10] (see also [9]), the fractional derivative operation $D_{0,z}^{\lambda,\mu,\eta}$ of a function $f(z)$ is defined as follows.

Definition 2. For $m-1\le\lambda<m; m\in N\ and\ \mu,\eta\in R$

(2.1) $$D_{0,z}^{\lambda,\mu,\eta} f(z)=\frac{d^m}{dz^m}\{\frac{z^{\lambda-\mu}}{\Gamma(m-\lambda)}\int_0^z (z-t)^{m-\lambda-1}\ {}_2F_1(\mu-\lambda,m-\eta;m-\lambda;1-\tfrac{t}{z})f(t)dt\},$$

where the function $f(z)$ is analytic in a simple connected region of the z-plane containing the region, with order

(2.2) $$f(z)=o(|z|^r),\ z\to 0,$$

where $r>\max\{0,\mu-\eta\}-1$ and the multiplicity of $(z-t)^{m-\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t)>0$ and is well defined in the unit disk.

The operator defined by (2.1) includes the known Riemann-Liouville fractional derivative operator $D_{0,z}^{\lambda,\mu,\eta}\ f(z)$. Indeed we have

(2.3) $$D_{0,z}^{\lambda,\mu,\eta}\ f(z)=D_{0,z}^{\lambda}\ f(z),$$

It is convenient to introduce here the fractional operator $J_{0,z}^{\lambda,\mu,\eta}$ which is defined in term of $D_{0,z}^{\lambda,\mu,\eta}$ as follows .

(2.4) $$J_{0,z}^{\lambda,\mu,\eta} \; f(z) = \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1-\mu+\eta)} z^{\mu} \; D_{0,z}^{\lambda,\mu,\eta} \; f(z) \quad ,$$

$$(\lambda \geq 0; \mu < p+1; \eta > \max\{\lambda, \mu\} - (p+1))$$

It may noted that if $\lambda = \mu$ in (2.4), then by virtue of (2.3) we have

(2.5) $$J_{0,z}^{\lambda,\mu,\eta} \; f(z) = \Gamma(2-\lambda) z^{\lambda} \; D_{0,z}^{\lambda} \; f(z)$$

and for $\lambda = \mu = 0$

(2.6) $$J_{0,z}^{0,0,\eta} \; f(z) = f(z)$$

also for $\lambda = \mu = 1$

(2.7) $$J_{0,z}^{1,1,\eta} \; f(z) = z \; f'(z)$$

Before starting and proving our main theorems, we need the following lemma to be used in the sequel (cf.Raina and Nahar [10]).

Lemma 1. If $(\lambda \geq 0; n > \max\{0, \mu-\eta\} - 1)$, then

(2.8) $$D_{0,z}^{\lambda,\mu,\eta} z^{n} = \frac{\Gamma(n+1)\Gamma(n-\mu+\eta+1)}{\Gamma(n-\mu+1)\Gamma(n-\lambda+\eta+1)} z^{n-\mu}$$

Coefficient estimates

Theorem 1: A function $f(z)$ defined by (1.2) belongs to the class $T_{p}^{n}(\alpha, \beta)$, $-1 \leq \alpha < p$, *and* $\beta \geq 0$ if and only if

(3.1) $$\sum_{n=p+1}^{\infty} \left(\frac{p(n-p)(1+\beta)}{m} - (\alpha - p) \right) B_m(n) \, a_n \leq (p-\alpha) \, ,$$

where

(3.2) $$B_m(n) = \frac{(m)_{n-p}(a)_{n-p}}{(1)_{n-p}(c)_{n-p}} \qquad a \geq c > 0 \quad , \; m \geq 1$$

and the result is sharp .

Proof. It is clear from (1.12) and (1.13) that

(3.3) $L_p(m+1,1)L_p(a,c)\ f(z)) = z^p - \sum_{n=p+1}^{\infty} \frac{(m+1)_{n-p}}{(1)_{n-p}(c)_{n-p}} a_n z^n$

$$= z^p - \sum_{n=p+1}^{\infty} \frac{m+n-p}{m} B_m(n)\, a_n z^n \quad .$$

where $B_m(n)$ is given by (3.2).

Assuming that (3.1) holds , then it suffices to show that

$$\beta\left|\frac{pL_p(m+1,1)(L_p(a,c)f(z))}{L_p(m+1,1)(L_p(a,c)f(z))} - p\right| - \mathrm{Re}\left\{\frac{pL_p(m+1,1)(L_p(a,c)f(z))}{L_p(m+1,1)(L_p(a,c)f(z))} - p\right\}$$

$$\leq (p-\alpha)$$

We have

$$\beta\left|\frac{pL_p(m+1,1)(L_p(a,c)f(z))}{L_p(m+1,1)(L_p(a,c)f(z))} - p\right| - \mathrm{Re}\left\{\frac{pL_p(m+1,1)(L_p(a,c)f(z))}{L_p(m+1,1)(L_p(a,c)f(z))} - p\right\}$$

$$\leq p(1+\beta)\left|\frac{L_p(m+1,1)(L_p(a,c)f(z))}{L_p(m+1,1)(L_p(a,c)f(z))} - 1\right|$$

$$\leq \frac{p(1+\beta)\sum_{n=p+1}^{\infty}\frac{n-p}{m}B_m(n)\,a_n}{1-\sum_{n=p+1}^{\infty}B_m(n)\,a_n} \quad ,$$

This expression is bounded above by $(p-\alpha)$ if

$$\sum_{n=p+1}^{\infty}\left(\frac{p(n-p)(1+\beta)}{m} - (\alpha-p)\right)B_m(n)\,a_n \leq (p-\alpha)\,.$$

Conversely, assume that $f(z)$ is in the class $T_p^n\ (\alpha,\beta)$, and z is real then we have from (1.14) and (3.3)

$$\frac{p-\sum_{n=p+1}^{\infty}\frac{m+n-p}{m}\,a_n\,z^{n-p}}{1-\sum_{n=p+1}^{\infty}B_m(n)\,a_n z^{n-p}}-\alpha\geq\beta\left|\frac{\sum_{n=p+1}^{\infty}\frac{p(n-p)}{m}B_m(n)\,a_n\,z^{n-p}}{1-\sum_{n=p+1}^{\infty}B_m(n)\,a_n\,z^{n-p}}\right|$$

Letting $z\to 1$ along the real axis, we obtain the desire inequality (3.1).
The equality in (3.1) is attained for the function

(3.4) $$f(z)=z^p-\frac{m(p-\alpha)}{[p(n-p)(1+\beta)-m(\alpha-p)]B_m(n)}z^n .$$

Corollary 1. Let the function $f(z)$ defined by (1.2) be in the class $T_p^n\ (\alpha,\beta)$ $\ -1\leq\alpha<p\,and\,\beta\geq 0.$ Then

$$\sum_{n=p+1}^{\infty}a_n\leq\frac{c(p-\alpha)}{a[p(1+\beta)-m(\alpha-p)]}\quad .$$

Distortion theorem

Theorem 2. Let $\lambda,\mu,\eta\in R$ such that $\lambda\geq 0;\mu<p+1;\eta>\max\{\lambda,\mu\}-(p+1)$, and let the function $f(z)$ defined be in be class $T_p^n\ (\alpha,\beta)$ .Then

(4.1) $$\left|D_{0,z}^{\lambda,\mu,\eta}f(z)\right|\geq\frac{\Gamma(p+1-\mu+\eta)}{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}|z|^{p-\mu}\quad\times$$

$$\left[1-\frac{2c(p-\alpha)(p+1+\eta-\mu)}{a[p(1+\beta)-m(\alpha-p)](p+1-\mu)(p+1+\eta-\lambda)}|z|\right],$$

and

(4.2) $$\left|D_{0,z}^{\lambda,\mu,\eta} f(z)\right| \le \frac{\Gamma(p+1-\mu+\eta)}{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)} |z|^{p-\mu} \times$$

$$\left[1+\frac{2c(p-\alpha)(p+1+\eta-\mu)}{a[p(1+\beta)-m(\alpha-p)](p+1-\mu)(p+1+\eta-\lambda)}|z|\right],$$

for $z \in U$ *and* $\mu \le 1$.

For $z \in U \ \backslash\{0\}$ and $\mu > 1$ the equalities in (4.1) and (4.2) are attained by the function given by (3.4).

Proof. Using the definition of fractional operator $J_{0,z}^{\lambda,\mu,\eta} f(z)$ defined in (2.4), and Lemma 1 we have

(4.3) $$J_{0,z}^{\lambda,\mu,\eta} f(z) = z^p - \sum_{n=p+1}^{\infty} \delta(n) a_n z^n ,$$

where

(4.4) $$\delta(n) = \frac{(p+1)_{n-p}(p+1-\mu+\eta)_{n-p}}{(p+1-\mu)_{n-p}(p+1-\lambda+\eta)_{n-p}} \qquad (n \ge p+1)$$

Under the conditions stated in the theorem, we observe that the function $\delta(n)$ is non-increasing, that is , it satisfies the inequality $\delta(n+1) \le \delta(n)$ for all $n \ge p+1$, and thus we have

(4.5) $$0 < \delta(n) \le \delta(p+1) = \frac{(p+1)\ (p+1-\mu+\eta)}{(p+1-\mu\ )(p+1-\lambda+\eta)}$$

Making use of (4.5) and corollary 1 in (4.3), we see that

$$\left| J_{0,z}^{\lambda,\mu,\eta} f(z) \right| \ge |z|^p - |z|^{p+1} \sum_{n=p+1}^{\infty} a_n \delta(n)$$

$$\left| J_{0,z}^{\lambda,\mu,\eta} f(z) \right| \ge |z|^p - |z|^{p+1} \delta(n) \sum_{n=p+1}^{\infty} a_n$$

$$\ge |z|^p - |z|^{p+1} \frac{m(p-\alpha)\delta(p+1)}{[p(1+\beta)-m(\alpha-p)]B_m}$$

$$\geq |z|^p - |z|^{p+1} \frac{m(p-\alpha)\ \delta(p+1)}{a[p(1+\beta)-m(\alpha-p)]}$$

$$\geq |z|^p \left[1 - \frac{m(p-\alpha)\delta(p+1)}{a[p(1+\beta)-m(\alpha-p)]}|z|\right],$$

Hence,

$$\left|D_{0,z}^{\lambda,\mu,\eta} f(z)\right| \geq \frac{\Gamma(p+1-\mu+\eta)}{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}|z|^{p-\mu} \quad \times$$

$$\left[1 - \frac{2c(p-\alpha)(p+1+\eta-\mu)}{a[p(1+\beta)-m(\alpha-p)](p+1-\mu)(p+1+\eta-\lambda)}|z|\right]$$

This is the assertion (4.1).
The assertion (4.2) can be proved similarly.

Corollary 2. Let the function $f(z)$ defined by (1.2) be in the class $T_p^n\ (\alpha,\beta)$. Then

$$|f(z)| \geq |z|^p \left[1 - \frac{c(p-\alpha)}{a[p(1+\beta)-m(\alpha-p)]}|z|\right], \tag{4.6}$$

and

$$|f(z)| \leq |z|^p \left[1 + \frac{c(p-\alpha)}{a[p(1+\beta)-m(\alpha-p)]}|z|\right]. \tag{4.7}$$

Proof. Setting $\lambda = \mu = 0, and\ n = p+1$ in theorem 2, using the relation in (2.6) we get the result.

Corollary 3. Let the function $f(z)$ defined by (1.2) be in the class $T_p^n\ (\alpha,\beta)$ .Then

$$|f'(z)| \geq |z|^{p-1} \left[1 - \frac{2\ c(p-\alpha)}{a[p(1+\beta)-m(\alpha-p)]}|z|\right], \tag{4.8}$$

and

(4.9) $$|f'(z)| \leq |z|^{p-1}\left[1+\frac{2c(p-\alpha)}{a[p(1+\beta)-m(\alpha-p)]}|z|\right].$$

Proof. Setting $\lambda=\mu=1$ in theorem 2, using the relation in (2.7) we get the result.

Radius of convexity for the class $T_p^n\ (\alpha,\beta)$.

Theorem 3. Let the function $f(z)$ defined by (1.2) be in the class $T_p^n\ (\alpha,\beta)$. Then $f(z)$ is convex in the disk $|z|<r_1=r_1(\alpha,\beta)$, where

(5.1) $$r_1(\alpha,\beta)=\inf_{n\geq p+1}\left\{\frac{p^2[p(n-p)(1+\beta)-m(\alpha-p)]}{mn^2(p-\alpha)}B_m(n)\right\}^{\frac{1}{n-p}}$$

The result is sharp for the function $f(z)$ defined by (3.4).

Proof**.** To establish the required result it is sufficient to show that

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right|\leq p, \quad for|z|<r_1$$

or equivalently

$$\frac{\sum_{n=p+1}^{\infty}n(n-p)a_n|z|^{n-p}}{p-\sum_{n=p+1}^{\infty}na_n|z|^{n-p}}\leq p\ ,$$

which is equivalent to show that

(5.2) $$\sum_{n=p+1}^{\infty}\left(\frac{n^2}{p^2}\right)a_n|z|^{n-p}\leq 1\ ,$$

as $f(z)\in T_p^n(\alpha,\beta)$, we have from Theorem 1

(5.3) $$\sum_{n=p+1}^{\infty}\frac{[p(n-p)(1+\beta)-m(\alpha-p)]B_m(n)a_n}{m(p-\alpha)}\leq 1,$$

Thus (5.2) is true if

$$(5.4) \qquad \frac{n^2}{p^2}|z|^{n-p} \le \left\{\frac{[p(n-p)(1+\beta)-m(\alpha-p)]}{m(p-\alpha)}B_m(n)\right\} .$$

Setting $|z|=r_1$ in (5.4) and simplify we get the result.

Integral transforms

Theorem 4. Let the function $f(z)$ defined by (1.2) be the class $T_p^n(\alpha,\beta)$. Then the integral transform

$$(6.1) \qquad F(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)\,dt \qquad c>-1$$

belong to $T_p^n(\alpha,\beta)$.

Proof. Using (1.2) and (6.1) we get

$$F(z)=z^p-\sum_{n=p+1}^{\infty}\frac{p+c}{n+c}a_n z^n$$

Therefore

$$\sum_{n=p+1}^{\infty}\left(\frac{p(n-p)(1+\beta)}{m}-(\alpha-p)\right)B_m(n)\left(\frac{c+p}{c+n}\right)a_n$$

$$\le\sum_{n=p+1}^{\infty}\left(\frac{p(n-p)(1+\beta)}{m}-(\alpha-p)\right)B_m(n)\,a_n$$

$$\le(p-\alpha).$$

which implies $F(z)\in T_p^n(\alpha,\beta)$.

Closure properties

Theorem 5 . Let the functions

$f_j(z)=z^p-\sum_{n=p+1}^{\infty}a_{nj}z^n \quad ,(j=1,,2,3,...,m)$ be in the class $T_p^n(\alpha,\beta)$.

Then the function $h(z)$ defined

by $h(z)=z^p-\sum_{n=p+1}^{\infty} d_n z^n$,belongs to $T_p^n(\alpha,\beta)$. where

$$d_n=\frac{1}{m}\sum_{n=p+1}^{\infty} a_{nj} \qquad (a_{nj}\ge 0)$$

Proof. Since $f_j(z)\in T_p^n(\alpha,\beta)$, it follows from Theorem 1, that

$$\sum_{n=p+1}^{\infty}\left(\frac{p(n-p)(1+\beta)}{m}-(\alpha-p)\right)B_m(n)\,a_n\le(p-\alpha), \tag{7.1}$$

Therefore

$$\sum_{n=p+1}^{\infty}\left(\frac{p(n-p)(1+\beta)}{m}-(\alpha-p)\right)B_m(n)\ d_n$$

$$=\sum_{n=p+1}^{\infty}\left(\frac{p(n-p)(1+\beta)}{m}-(\alpha-p)\right)B_m(n)\left(\frac{1}{m}\sum_{j=2}^{m}a_{nj}\right)$$
$$\le(p-\alpha)$$

by (7.1) , which shows that $h(z)\in T_p^n(\alpha,\beta)$.

Theorem 6. Let $f_1(z)=z^p$ and

$$f_n(z)=z^p-\frac{m(p-\alpha)}{[p(n-p)(1+\beta)-m(\alpha-p)]B_m(n)}z^n \ , (n\ge p+1) \tag{7.2}$$

Then $f(z)\in T_p^n(\alpha,\beta)$. iff it can be expressed in the form

$$f(z)=\lambda_1\ f_1(z)+\sum_{n=p+1}^{\infty}\lambda_n\ f_n(z)\ , \tag{7.3}$$

where $\lambda_n\ge 0$ *and* $\sum_{n=p}^{\infty}\lambda_n=1$.

Proof . Let (7.3) hold, then by (7.2) we have

$$f(z)=z^p-\sum_{n=p=1}^{\infty}\frac{m(p-\alpha)}{[p(n-p)(1+\beta)-m(\alpha-p)]B_m(n)}\lambda_n z^n \ .$$

Now

$$\sum_{n=p+1}^{\infty}\frac{1}{m}[p(n-p)(1+\beta)-m(\alpha-p)]B_m(n)\,a_n$$

$$= \sum_{n=p+1}^{\infty} \frac{1}{m}[p(n-p)(1+\beta)-m(\alpha-p)]B_m(n) \quad \times$$

$$\frac{m(p-\alpha)}{[p(n-p)(1+\beta)-m(\alpha-p)]B_m(n)}\lambda_n$$

$$=(p-\alpha)\sum_{n=p+1}^{\infty}\lambda_n$$

$$\leq (p-\alpha)\sum_{n=p}^{\infty}\lambda_n \leq (p-\alpha) \ .$$

Hence by Theorem 1, $f(z)\in T_p^n(\alpha,\beta)$.

Conversely, suppose $f(z)\in T_p^n(\alpha,\beta)$. Since

$$a_n \leq \frac{m(p-\alpha)}{[p(n-p)(1+\beta)-m(\alpha-p)]B_m(n)} \quad , \qquad (n\geq p+1)$$

Setting $\lambda_n = \dfrac{[p(n-p)(1+\beta)-m(\alpha-p)]}{m(p-\alpha)}a_n$

and $\lambda_1 = 1 - \sum_{n=p+1}^{\infty}\lambda_n$, we get (7.3).

This completes the proof of Theorem.

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