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$b-$ Open sets and Locally Sierpinski spaces on T_o-Alexandroff spaces

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***Abstract**: The aim of this paper is to continue the study of T_o-Alexandroff spaces, a class of topological spaces which lies strictly between classes of Alexandroff spaces and locally finite spaces. We study b-open sets and prove that BO(X)=SO(X) in Artinian T_o-Alexandroff spaces. After that, we investigate locally Sierpinski spaces. Each locally Sierpinski space is submaximal T_o-Alexandroff space. The converse is not true.*

In this paper we will introduce a sufficient condition for a submaximal T_o-Alexandroff space to be locally Sierpinski space.

***Keywords**:* Alexandroff spaces, Artinian T_o-Alexandroff spaces, b-open sets, semi-open sets, locally Sierpinski spaces, submaximal.

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1. INTRODUCTION

By the *Alexandroff space*, we mean a topological space X in which the intersection of any family of open sets is open and the union of any number of closed sets is closed. Equivalently, every point x in X has a minimal neighborhood – denoted by $V(x)$ – being the intersection of all open sets containing x 1. So all finite spaces are Alexandroff spaces.

We focus on Alexandroff spaces that satisfy the separation axiom T_0, because there exists a functorial equivalent between the categories of

T_0-Alexandroff spaces and partially ordered sets where each one of them is completely determined by the other.

Given a T_0-Alexandroff space (X,τ), the partial order $\leq_\tau$ - called (*Alexandroff*) *specialization order* - is defined by $a \leq_\tau b$ iff $a \in \overline{\{b\}}$.

We denote a T_o−Alexandroff space with its specialization order to be the pair $(X, \tau(\leq))$ where the corresponding poset is $(X, \leq)$.

These spaces are related to the study of digital topology. The interest in Alexandroff spaces is a consequence of the very important role of finite spaces in digital topology.

Many results and identifications related to open sets and generalized open sets were introduced in some of our previous published papers about T_o−Alexandroff. More precisely, in these papers, we characterized basic concepts of open sets, topological properties, generalized open sets such as preopen sets, semi-open sets and α-open sets. We proved that $\tau_\alpha = PO(X)$ and $\tau_\alpha \subseteq SO(X)$. We introduced two subclasses of T_o−Alexandroff spaces called Artinian and Noetherian T_o−Alexandroff spaces.

Later, lower separation axioms related to open and generalized open sets such as T_i, $i = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}$ and $semi-T_i$, $\alpha - T_i$ $i = \frac{1}{2}, 1, 2$ where studied and characterized also. For more details, see 10, 15, 16, and 17.

This paper is a continuation of the previous formalist of our studies on T_o−Alexandroff spaces. We study *b*-open sets and prove that $BO(X)=SO(X)$ in Artinian T_o−Alexandroff spaces. After that we investigate locally Sierpinski spaces. Each locally Sierpinski space is submaximal T_o−Alexandroff space. The converse is not true. In this paper we will introduce a sufficient condition for a submaximal T_o−Alexandroff space to be locally Sierpinski space.

For complete study, the reader should be familiar with our previous studies and intuitive concepts of Alexandroff spaces.

Throughout this paper, the symbol $(X, \tau(\leq))$ denotes a T_0–Alexandroff space equipped with its (Alexandroff) specialization order $\leq$. For each element $x \in X$, $\uparrow x = \{y : x \leq y\}$ is the minimal neighborhood of x . For a subset A of X the interior (resp. the closure, the boundary, the semi-interior, the semi-closure, the preinterior, the preclosure, the $b-$interior, the $b-$closure) will be denoted by A^o (resp. , $\overline{A}, bd(A)$, *sInt*(*A*), *sCl*(*A*), *pInt*(*A*), *pCl*(*A*), *bInt*(*A*), *bCl*(*A*)).

2. Definitions and Preliminaries

Preopen, semi-open, $\alpha-$open, *b*-open, and some other concepts of generalized open sets have been considered. The name preopen was used for the first time by Mashhour, abd El-Monsef , and El-Deeb 19 The definition of a preopen was introduced by Corson and Michael 5 under the name locally dense (see 5, 14, and 20) .

2.1. Definitions. A subset *A* of a space *X* is called

(i) *semi-open set 14* if $A \subseteq \overline{A^o}$, and *semi-closed set* 6 if A^c is semi-open. Thus *A* is semi-closed if and only if $\overline{A}^o \subseteq A$. If *A* is both semi-open and semi-closed then *A* is called semi-regular 18,

(ii) *preopen set* 19 if $A \subseteq \overline{A}^o$, and *pre-closed set* 11 if A^c is preopen. Thus *A* is preclosed if and only if $\overline{A^o} \subseteq A$,

(iii) *α-open set* 20 if $A \subseteq \overline{A^o}^o$, and *α-closed set* 8 if A^c is α-open. Thus *A* is α-closed if and only if $\overline{\overline{A}^o} \subseteq A$.

The family of all semi-open (resp. preopen, α-open) sets is denoted by *SO*(*X*) (resp. *PO*(*X*), τ_α). Njåstad 20 proved that τ_α is a topology on *X*. In general, *SO*(*X*) and *PO*(*X*) need not be topologies on *X*. If $A \subseteq$

X then $pInt(A)$ (resp. $sInt(A)$) is the largest preopen set (resp. semi-open set) inside A. $pCl(A)$ (resp. $sCl(A)$) is the smallest preclosed set (resp. semi-closed set) contains A.

2.2. Definitions.

1. A poset $(X, \leq)$ satisfies
   (i) *the ascending chain condition* (briefly, *ACC*) if each increasing chain is finally constant.
   (ii) *the descending chain condition* (briefly, *DCC*) if each decreasing chain is finally constant.
2. If $(X, \tau(\leq))$ is a T_o–Alexandroff space equipped with its specialization order. Then
   (i) X is called an *Artinian T_o–Alexandroff space* 16, if the corresponding poset $(X, \leq)$ satisfies the *ACC*.
   (ii) X is called a *Noetherian T_o–Alexandroff space*16, if the corresponding poset $(X, \leq)$ satisfies the *DCC*.
   (iii) if X is both Artinian and Noetherian T_o–Alexandroff space, then X is called *a generalized locally finite* (briefly, *g-LF*).

If (X, τ) is a topological space. We denote as $\overline{x}$ the closure of the point $x \in X$. The points with the property $\overline{x} = x$ are called *c-vertices*. The singular points x (= the open set points) are called *o-vertices*. If x is both *c*-vertex and *o*-vertex, then x is called *a pure element*.

If $(X, \tau(\leq))$ is a T_o–Alexandroff space equipped with its specialization order, then x is *o*-vertex (resp. *c*-vertex) iff x is maximal (resp. minimal) point in the corresponding poset.

2.3. Definitions.*16* *If X is an Artinian (resp. Noetherian) T_o−Alexandroff space equipped with its specialization order, then we define*

(i) M (resp. m) to be the set of all maximal (resp. minimal) elements in X.

(ii) for $A \subseteq X$, $M(A)$ (*resp.* $m(A)$) to be the set of all maximal (resp. minimal) element in A with respect to the induced order.

(iii) for $x \in X$, $\hat{x}$ (resp. $\check{x}$) is the set of all maximal elements in X greater than (resp. less than) or equal to x. Clearly that $\hat{x} = \uparrow x \cap M$ and $\check{x} = \downarrow x \cap m$. Moreover, x is *o*-vertex if $\{x\} = \uparrow x$ and x is *c*-vertex if $\{x\} = \downarrow x$.

2.4. Definition *A space (X, τ) is called a submaximal 12 if each dense subset is open.*

2.5. Proposition.*16* *Let $(X, \tau(\leq))$ be an Artinian T_o−Alexandroff space. Then a set A is semi-open if and only if $M(A) \subseteq M$.*

2.6. Proposition.16 *Let $(X, \tau(\leq))$ be an Artinian T_o−Alexandroff space and let $A \subseteq X$. Then $pInt(A) \subseteq sInt(A)$ and $sCl(A) \subseteq pCl(A)$.*

2.7. Definitions. The topological space (X, τ) is

(i) a $T_{\frac{1}{2}}$ −*space* 13 if every singleton is either open or closed 9.

(ii) a $T_{\frac{3}{4}}$ −*space* 7 if every singleton is either regular open or closed.

3. b − OPEN SETS IN T_o−ALEXANDROFF SPACES

3.1 Definitions. 2 A subset A of a space (X, τ) is called b-open if $A \subseteq \overline{A^o} \cup \overline{A}^o$. A complement of b-open set is *b-closed*. The class of all b-open sets in X will be denoted by $BO(X)$. Let $A \subseteq X$ then $bInt(A)$ is the largest b-open set inside A, and $bCl(A)$ is the smallest b-closed set contains A.

It is obvious that $PO(X) \bigcup SO(X) \subseteq BO(X)$.

3.2 Proposition.2 *For a subset A of a space X, the following are equivalent:*

(i) A *is* b-*open.*

(ii) $A = pInt(A) \bigcup sInt(A)$.

(iii) $A \subseteq pCl(pInt(A))$.

3.3 Proposition. 2 *For a subset A of a space X, we have*

(i) $bInt(bCl(A)) = bCl(bInt(A))$.

(ii) $bCl(A) = sCl(A) \bigcap pCl(A)$.

(iii) $bInt(A) = sInt(A) \bigcup pInt(A)$.

3.4 Definition. Let $f : X \to Y$ be a function from a topological space X into a topological space Y, we say that

a) f is *b-continuous* (*semi-continuous*) *function* if the inverse image of each open set in Y is *b*-open (semi-open).

b) f is *b-open* (*semi-open*) *function* if the image of each *b*-open (semi-open) set in X is open in Y.

3.5 Proposition. *Let X be an Artinian T_o−Alexandroff space and let $A \subseteq X$. A is semi-open iff A is b-open; that is, the two classes SO(X) and BO(X) are identical.*

Proof. The set A is b-open iff $A = pInt(A) \bigcup sInt(A)$. So, by Proposition 2.6 A is b-open iff $A=sInt(A)$ iff A is semi-open set.

3.6 Corollary. *Let X be an Artinian T_o−Alexandroff space. For any $A \subseteq X$, $sInt(sCl(A)) = sCl(sInt(A))$.*

3.7 Corollary. *Let $f : X \to Y$ be a function from an Artinian T_o−Alexandroff space X into a topological space Y. Then*

a) *f is b-continuous iff f is semi-continuous.*

b) *f is b-open iff f is semi-open.*

In 16, we give a characterization of a semi-open set and some of its related properties on Artinian T_o-Alexandroff space in the sense of its corresponding poset. Needless to say that, we can carry the results of semi-open sets to b-open sets to get new results. More precisely, if X is an Artinian T_o-Alexandroff space and if $A \subseteq X$, then

1. A is *b*-open set iff $M(A) \subseteq M$.
2. A is *b*-closed set iff $\forall x \notin A,\ x \not\subset A$.
3. $bCl(A) \subseteq pCl(A)$.
4. $pInt(A) \subseteq bInt(A)$.
5. $bInt(A) = \{x \in A : x \cap A \neq \emptyset\}$.
6. $bCl(A) = A \cup \{x : x \subseteq A\}$.

Recall that none of the classes *SO*(*X*), *PO*(*X*), and *BO*(*X*) is a topology on *X*, and these classes generates a topology on *X*. Suppose that ξ stands for *SO*(*X*), *PO*(*X*), and *BO*(*X*), we define a topology generated by ξ to be

$$\tau(\xi) = \{U \subseteq X : U \cap A \in \xi, \quad \forall A \in \xi\}$$

The fact that $\tau(\xi)$ is a topology on *X* larger than τ was studied. Njåsted 20 showed that $\tau(SO(X)) = \tau_\alpha$. The topology generated by *PO*(*X*) (resp. *BO*(*X*)) is denoted by τ_γ 3 (resp. τ_b 2). Andrijević in 2 showed that $\tau_b = \tau_\gamma$. These facts raise the following proposition.

3.8 Proposition. *Let* $(X, \tau\ (\leq))$ *be an Artinian* T_o-*Alexandroff space. Then* $\tau_\alpha = \tau_\gamma = \tau_b$.

This proposition agrees with our result in 16; that is, in Artinian T_o-Alexandroff spaces, $\tau_\alpha = PO(X) = \tau_\gamma$.

4. LOCALLY SIERPINSKI SPACES

The objective now is to concentrate our attention on a class of topological spaces, called locally Sierpinski spaces. As we show below, this class lies strictly inside the class of finite chain spaces.

4.1 Definition.4 A topological space X is called *locally Sierpinski space* (briefly, *LS-space*) if every point $x \in X$ has a neighborhood homeomorphic to the Sierpinski space.

It is well known that X is *LS*-space if for each $x \in X$ **,** there exists an open set $S_x = \{x, y\}$ such that the relative topology on S_x is a copy of Sierpinski topology. This set is called *Sierpinski set*. It is evident that a *LS*-space is T_o-Alexandroff space.

4.2 Proposition.21 *Let X be a LS-space equipped with its specialization order. Then for each* $x \in X$, *x is either maximal or minimal element in the corresponding poset.*

As a consequence of Proposition 4.2, the class of *LS*-spaces is proper subclass of *g-LF* spaces.

Recall that a topological space X is called locally finite if each element x of X is contained in a finite open set and a finite closed set. It is well known that locally finite spaces are Alexandroff. So, every T_o–locally finite space is *g-LF*. At the first glimpse, it is easy to believe the converse, but the following example shows that the converse is not true.

4.3 Example. Let $X = \mathbb{N} \cup \{\mathrm{T}\}$ with anti-chain order on $\mathbb{N}$ and for each $x \in \mathbb{N}$, $x \leq \mathrm{T}$.(see Figure 1 below)

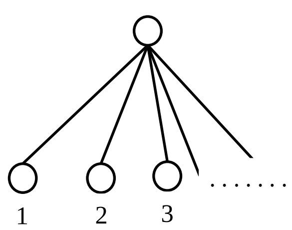

FIGURE 1

It is clearly that X is *g-LF* and it is not locally finite, since there is no finite closed set containing the element T. Also, note that X is *LS*-space.

4.4 Proposition.15 *Let $(X,\tau\ (\leq))$ be a T_o-Alexandroff space. Then the following are equivalent:*

1. *X is $T_{\frac{1}{2}}-$space.*
2. *Each element of X is pure; that is, every singleton is either maximal or minimal.*
3. *X is submaximal.*
4. *$PO(X)=\tau_{\alpha}=\tau$; that is, every preopen set is open*

4.5 Theorem.15 *Let $(X,\tau\ (\leq))$ be a T_o-Alexandroff space. Then X is $T_{\frac{3}{4}}$ if and only if the following two conditions are satisfied*

(a) *X is $T_{\frac{1}{2}}-$space,*

(b) $\forall x \notin M$, $|\hat{x}| \geq 2$.

As a consequence of Proposition 4.2 and Proposition 4.4, if X is *LS*-space, then X is submaximal and $T_{\frac{1}{2}}-$space. Moreover, $PO(X) = \tau_{\alpha} = \tau$. The converse is not true; that is, not all submaximal spaces are locally Sierpinski. Simply, let $X=\{a,b,c\}$ with minimal base $B=\{\{a\},\{b\},X\}$. Then X is submaximal space which is not *LS*-space.

We deal with the question under what sufficient conditions a submaximal space satisfy to be *LS*-Space? The following proposition is the main result in this section.

4.6 Proposition. A submaximal space X is *LS*-space iff the two conditions hold:

(a) $\forall x \notin M$, $|\hat{x}|=1$.

(b) There is no clopen (= closed and open) singleton sets.

Proof. $(\Rightarrow)$ If there exists $x \notin M$ such that $|\hat{x}| \geq 2$. Then for each $y \in X-\{x\}$, the set $\{x,y\}$ is not open and hence not Sierpinski set.

If $\{x\}$ is clopen singleton set, we have two cases for elements $y \in X - \{x\}$:

Case1: y is isolated point, in this case, the set $\{x,y\}$ has discrete relative topology.

Case2: {y} is closed set and not open, in this case, the set $\{x, y\}$ is not open and hence not Sierpinski set.

$(\Leftarrow)$ Let $x \in X$. If x is isolated point, by condition (b), there exists $y \in X - \{x\}$ such that $y \leq x$. Moreover, by condition (a), $|y|$=1. This implies that the set $S_x=\{x,y\}$ is open Sierpinski set.

Finally, if x is not isolated, by condition (a), $x = \{y\}$ for some $y \neq x$. Hence $S_x=\{x,y\}$ is open Sierpinski set $\square$

4.7 Remark. Informally, If we think for a partial order $\leq$ on a *LS*-space as a relation between two sets; the set of minimal elements *m* and the set of maximal elements *M*, then the relation $\leq$ becomes a function from the domain *m* onto the range *M*. This gives a nice way to investigate a given T_o−Alexandroff space if it is *LS*-space.

From the remark above, we establish that $|m| \geq |M|$. This result and Proposition 4.6 part (b) give directly the following result in 21 as a corollary.

4.8 Corollary. If X is a finite *LS*-space, and if $|X| = n$, then the number of isolated points can not exceed $\left[\frac{n}{2}\right]$ the integer part of $\frac{n}{2}$.

4.9 Corollary. *A topological space X can not be both LS-space and $T_{\frac{3}{4}}$-space; that is, the property 'space being LS' can not appear together with the property 'space satisfies the separation axiom $T_{\frac{3}{4}}$'.*

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