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Near rough probability in topologized stochastic approximation spaces

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Abstract: *Most researches in rough set context are based on Pawlak's model and its generalization. Topology was used as a tool for rough set extension. The purpose of this paper is to generalize approximation spaces using topology and probability theory. Pawlak's approach about rough probability and a generalization of Pawlak's approach are given. In this paper we introduce near rough probability in approximation spaces.*

Keywords: *Topological space; Topologized stochastic approximation space; Near rough probability*

1. Introduction

The concept of topological structures and their generalizations are one of the most powerful notions in system analysis [1]. Many works have appeared recently for example in structural analysis, in chemistry, and physics [2, 3, 4]. The purpose of the present work is to put a starting point for the applications of abstract topological theory into rough set analysis. Rough set theory, introduced by Pawlak in 1982 [5], is a mathematical tool that supports also the uncertainty reasoning but qualitatively. In this paper, we shall integrate some ideas in terms of concepts in topology.

2. Preliminaries

This section presents a review of some fundamental notions of topological spaces and rough set theory.

A topological space [1] is a pair (X,τ) consisting of a set X and a family τ of subsets of X satisfying the following conditions:

(T1) $\phi \in \tau$ and $X \in \tau$.
(T2) τ is closed under arbitrary union.
(T3) τ is closed under finite intersection.

Throughout this paper (X,τ) denotes a topological space, the elements of X are called points of the space, the subsets of X belonging to τ are called open sets in the space, the complement of the subsets of X belonging to τ are called closed sets in the space, and the family of all $\tau-$closed subsets of X is denoted by τ^*, the family τ of open subsets of X is also called a topology for X.

A family $\mathrm{B} \subseteq \tau$ is called a base for (X,τ) iff every nonempty open subset of X can be represented as a union of subfamily of B. Clearly, a topological space can have many bases. A family $\mathrm{S} \subseteq \tau$ is called a subbase iff the family of all finite intersections of S is a base for (X,τ).

The τ-closure of a subset $A \subseteq X$ is denoted by A^- and is given by $A^- = \cap\{F \subseteq X : A \subseteq F \text{ and } F \in \tau^*\}$.
Evidently, A^- is the smallest closed subset of X which contains A. Note that A is closed iff $A = A^-$.
The τ-interior of a subset $A \subseteq X$ is denoted by A° and is given by $A^\circ = \cup\{G \subseteq X : G \subseteq A \text{ and } G \in \tau\}$. Evidently, A° is the union of all open subsets of X which contained in A. Note that A is open iff $A = A^\circ$.
Motivation for rough set theory has come from the need to represent subsets of a universe in terms of equivalence classes of a partition of that universe. The partition characterizes a topological space, called approximation space $K = (X,R)$, where X is a set called the universe and R is an equivalence relation [6, 7]. The equivalence classes of R are also known as the granules, elementary sets or blocks; we shall use $R_x \subseteq X$ to denote the equivalence class containing $x \in X$, and X/R to denote the set of all elementary sets of R. In the approximation space, we consider two operators, the upper and lower approximations of subsets: Let $A \subseteq X$, then the lower approximation (resp. the upper approximation) of A is given by

$$\underline{R}A = \{x \in X : R_x \subseteq A\}$$
$$\left(\text{resp. } \overline{R}A = \{x \in X : R_x \cap A \neq \phi\}\right).$$

Pawlak noted that [8] the approximation space $K = (X,R)$ with equivalence relation R defines a uniquely topological space (X,τ_K) where τ_K is the family of all clopen sets and X/R is a base of τ_K. Moreover the lower (resp. upper) approximation of any subset $A \subseteq X$ is exactly the interior (resp. closure) of the subset A. In this section we shall generalize Pawlak's concepts in the case of general relations. Hence the approximation space $K=(X,R)$ with general relation R defines a uniquely topological space (X,τ_K) where τ_K is the topology associated to K (i.e. τ_K is the family of all open sets in (X,τ_K) and $X/R=\{xR : x\in X\}$ is a subbase of τ_K, where $xR=\{y\in X : xRy\}$). We shall give this hypothesis in the following definition.

Definition 2.1. Let $K=(X,R)$ be an approximation space with general relation R and τ_K is the topology associated to K. Then the triple $\kappa=(X,R,\tau_K)$ is called a topologized approximation space.

The following definition introduces lower and upper approximations in a topologized approximation space $\kappa=(X,R,\tau_K)$.

Definition 2.2. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space. If $A\subseteq X$, then the lower approximation (resp. upper approximation) of A is defined by

$$\underline{R}A=A^{\circ} \text{ (resp. } \overline{R}A=A^{-}\text{)}.$$

3. Pawlak's approach

Consider the approximation space $K=(U,R)$, where U is a set called the universe and R is an equivalence relation. The order triple $S=(U,R,p)$ is called the stochastic approximation space [9], where p is a probability measure, any subset of U will be called an event. The probability measure p has the following properties:

$p(\phi)=0,\ p(U)=1$ and if $A=\bigcup_{i=1}^{n} X_i$ is an observable set in K, then $p(A)=\sum_{i=1}^{n} p(X_i)$.

It is clear that A is a union of disjoint sets, since R is an equivalence relation. Pawlak introduced the definitions of the lower and upper probabilities of an event A in the stochastic approximation space $S=(U,R,p)$. These definitions are:

- The lower probability of A is denoted by $\underline{p}(A)$ and is given by $\underline{p}(A)=p(\underline{R}A)$.
- The upper probability of A is denoted by $\overline{p}(A)$ and is given by $\overline{p}(A)=p(\overline{R}A)$.

Clearly, $0\le \underline{p}(A)\le 1$ and $0\le \overline{p}(A)\le 1$.

The probability measure p in the stochastic approximation space $S=(U,R,p)$ satisfies the following properties [9]:

1) $\underline{p}(A)\le p(A)\le \overline{p}(A)$.
2) $\underline{p}(\phi)=\overline{p}(\phi)=0$.
3) $\underline{p}(U)=\overline{p}(U)=1$.
4) $\underline{p}(A^c)=1-\overline{p}(A)$.
5) $\overline{p}(A^c)=1-\underline{p}(A)$.
6) $\overline{p}(A\cup B)\le \overline{p}(A)+\overline{p}(B)-\overline{p}(A\cap B)$
7) $\underline{p}(A\cup B)\ge \underline{p}(A)+\underline{p}(B)-\underline{p}(A\cap B)$

Definition 3.1 [9]. Let A be an event in the stochastic space $S=(U,R,p)$. Then the rough probability $p^*(A)=\langle \underline{p}(A),\overline{p}(A)\rangle$, where $\underline{p}(A)=p(\underline{R}A)$ and $\overline{p}(A)=p(\overline{R}A)$.

Clearly, the rough probability is the interval to which the probability of the unobservable event belongs. If A is an observable event in S, the rough probability $p^*(A)$ will be the same as the classical probability $p(A)$; that is, $p^*(A)$ will be reduced to one point.

Moreover,

- If A is externally unobservable, then $p^*(A)=\langle \underline{p}(A),1\rangle$.
- If A is internally unobservable, then $p^*(A)=\langle 0,\overline{p}(A)\rangle$.
- If A is totally unobservable, then $p^*(A)=\langle 0,1\rangle$.

An exact value of the probability of event A is given if it is observable, if A is roughly observable a lower and upper values to the probability of A is given. In the case when the event A is internally (resp. externally) unobservable, only the upper (resp. lower) bound can be determined. But if A is totally unobservable both the lower and upper bounds for the probability of A can not be determined.

4. Near rough probability in topological spaces

The present section is devoted to introduce the near rough probability by applying the concepts of near open sets. We study approximation spaces from topological view and obtain some rules to find lower and upper probabilities in several ways in approximation spaces with general relations. We shall recall some definitions about some classes of near open sets which are essential for our present study. Some forms of near open sets are introduced in the following definition.

Definition 4.1. Let (X,τ) be a topological space, then the subset $A \subseteq X$ is called:

i) Regular open [10] (briefly $r-$open) if $A = A^{-\circ}$.

ii) Semi-open [11] (briefly $s-$open) if $A \subseteq A^{\circ -}$.

iii) Pre-open [12] (briefly $\mathrm{p}-$open) if $A \subseteq A^{-\circ}$.

iv) γ -open [13] ($\mathrm{b}-$open [14]) if $A \subseteq A^{\circ -} \cup A^{-\circ}$.

v) α -open [15] if $A \subseteq A^{\circ - \circ}$.

vi) β -open [16] (semi-pre-open [17]) if $A \subseteq A^{-\circ -}$.

Notice 4.1.

i) The complement of an $r-$open (resp. $s-$open, $\mathrm{p}-$open, γ-open, α-open and β-open) set is called $r-$closed (resp. $s-$closed, $p-$closed , γ-closed, α-closed and β-closed) set .

ii) The family of all $r-$open (resp. $s-$open, $\mathrm{p}-$open, γ-open, α-open and β-open) sets of (X,τ) is denoted by $RO(X)$ (resp. $SO(X)$, $PO(X)$, $\gamma O(X)$, $\alpha O(X)$ and $\beta O(X)$).

iii) The family of all $r-$closed (resp. $s-$closed, $\mathrm{p}-$closed, γ- closed, α- closed and β-closed) sets of (X,τ) is denoted by $RC(X)$ (resp. $SC(X)$, $PC(X)$, $\gamma C(X)$, $\alpha C(X)$ and $\beta C(X)$).

The aim of the following example is to illustrate the existence of spaces in which the above classes of near open sets and near closed sets are not coincided and do not have the discrete structure.

Example 4.1. Let $X = \{a,b,c,d\}$ and $\tau = \{X,\phi,\{a,b\},\{d\},\{a,b,d\}\}$. Then the classes of near open sets are

$$RO(X) = \{X,\phi,\{d\},\{a,b\}\}$$
$$SO(X) = \{X,\phi,\{d\},\{a,b\},\{c,d\},\{a,b,c\},\{a,b,d\}\}$$
$$PO(X) = \{X,\phi,\{a\},\{b\},\{d\},\{a,b\},\{a,d\},\{b,d\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}$$
$$\gamma O(X) = \{X,\phi,\{a\},\{b\},\{d\},\{a,b\},\{a,d\},\{b,d\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}$$
$$\alpha O(X) = \{X,\phi,\{d\},\{a,b\},\{a,b,d\}\}$$
$$\beta O(X) = \{X,\phi,\{a\},\{b\},\{d\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}.$$

From known results [13, 16] we have the following two remarks. We shall use the symbol "$\xrightarrow{\subseteq}$" instead of "$\subseteq$" in the implications between sets.

Remark 4.1. In a topological space (X,τ), the implications between τ and the families of near open sets are given in the following diagram.

$$\begin{tikzcd}
 & & & SO(X) \arrow[dr, "\subseteq"] & & \\
RO(X) \arrow[r, "\subseteq"] & \tau \arrow[r, "\subseteq"] & \alpha O(X) \arrow[ur, "\subseteq"] \arrow[dr, "\subseteq"'] & & \gamma O(X) \arrow[r, "\subseteq"] & \beta O(X) \\
 & & & PO(X) \arrow[ur, "\subseteq"'] & &
\end{tikzcd}$$

Remark 4.2. In a topological space (X,τ), the implications between τ^* and the families of near closed sets are given in the following diagram.

$$\begin{tikzcd}
 & & & SC(X) \arrow[dr, "\subseteq"] & & \\
RC(X) \arrow[r, "\subseteq"] & \tau^* \arrow[r, "\subseteq"] & \alpha C(X) \arrow[ur, "\subseteq"] \arrow[dr, "\subseteq"'] & & \gamma C(X) \arrow[r, "\subseteq"] & \beta C(X) \\
 & & & PC(X) \arrow[ur, "\subseteq"'] & &
\end{tikzcd}$$

The following definition is given to introduce the near interior of a subset A of X in a topological space (X,τ).

Definition 4.2. Let (X,τ) be a topological space and $A \subseteq X$, then the near interior (briefly $j-$ interior) of A is denoted by $A^{j\circ}$ for all $j \in \{r, s, \mathrm{p}, \gamma, \alpha, \beta\}$ and is defined by

$$A^{j\circ} = \cup \{G \subseteq X : G \subseteq A,\ G \text{ is a } j-\text{open set}\}.$$

The aim of the following definition is to introduce the near closure of a subset A of X in a topological space (X,τ).

Definition 4.3. Let (X,τ) be a topological space and $A \subseteq X$, then the near closure (briefly $j-$ closure) of A is denoted by A^{j-} for all $j \in \{r, s, \mathrm{p}, \gamma, \alpha, \beta\}$ and is defined by

$$A^{j-} = \cap \{H \subseteq X : A \subseteq H,\ H \text{ is a } j-\text{closed set}\}.$$

The following two definitions introduce near lower and near upper approximations in a topologized approximation space $\kappa = (X, R, \tau_K)$.

Definition 4.4. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. If $A \subseteq X$, then the near lower approximation (briefly $j-$lower approximation) of A is denoted by $\underline{R}_j\, A$ and is defined by

$$\underline{R}_j\, A = A^{j\circ}, \text{ where } j \in \{r, s, \mathrm{p}, \gamma, \alpha, \beta\}.$$

Definition 4.5. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. If $A \subseteq X$, then the near upper approximation (briefly $j-$upper approximation) of A is denoted by $\overline{R}_j\, A$ and is defined by

$$\overline{R}_j\, A = A^{j-}, \text{ where } j \in \{r, s, \mathrm{p}, \gamma, \alpha, \beta\}.$$

Proposition 4.1 [18]. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. If $A \subseteq X$, then the implications between lower approximation and $j-$lower approximations of A are given by the following diagram for all $j \in \{s, \mathrm{p}, \gamma, \alpha, \beta\}$.

$$\begin{tikzcd}
 & & \underline{R}_s\, A \arrow[dr, "\subseteq"] & & \\
\underline{R}\, A \arrow[r, "\subseteq"] & \underline{R}_\alpha\, A \arrow[ur, "\subseteq"] \arrow[dr, "\subseteq"'] & & \underline{R}_\gamma\, A \arrow[r, "\subseteq"] & \underline{R}_\beta\, A \\
 & & \underline{R}_p\, A \arrow[ur, "\subseteq"'] & &
\end{tikzcd}$$

Proposition 4.2 [18]. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. If $A \subseteq X$, then the implications between upper approximation and $j-$upper approximations of A are given by the following diagram for all $j \in \{s, \mathrm{p}, \gamma, \alpha, \beta\}$.

$$\begin{tikzcd}
 & & \overline{R}_s\, A \arrow[dr, "\subseteq"] & & \\
\overline{R}_\beta\, A \arrow[r, "\subseteq"] & \overline{R}_\gamma\, A \arrow[ur, "\subseteq"] \arrow[dr, "\subseteq"'] & & \overline{R}_\alpha\, A \arrow[r, "\subseteq"] & \overline{R}\, A \\
 & & \overline{R}_p\, A \arrow[ur, "\subseteq"'] & &
\end{tikzcd}$$

4.1 Near rough probability

In this section we shall introduce the near rough (briefly $j-$rough) probability for all $j \in \{r, s, p, \gamma, \alpha, \beta\}$. We study stochastic approximation spaces from topological view and generalize the stochastic approximation space in the case of general relation. Since the approximation space $K=(U,R)$ with general relation R defines a uniquely topological space (U,τ_K) where τ_K is the family of all open sets in (U,τ_K) and U/R is a subbase of τ_K, then the order triple $S=(U,R,p)$ is called the stochastic approximation space, where R is a general relation and p is a probability measure. This hypothesis is introduced in the following definition.

Definition 4.1.1. Let $K=(U,R)$ be an approximation space with general relation R and τ_K is the topology associated to K. Then the order 4-tuple $\mathfrak{S}=(U,R,p,\tau_K)$ is called a topologized stochastic approximation space.

The probability measure p has the following properties:

$p(\phi)=0,\ \ p(U)=1$ and if $A=\bigcup_{i=1}^{n} X_i$ is an observable set in K, then

$$p(A)=\sum_{i=1} p(X_i)-\sum_{i<j} p(X_i\cap X_j)$$
$$+\sum_{i<j<k} p(X_i\cap X_j\cap X_k)-\ldots\pm p(X_1\cap\ldots\cap X_n).$$

It is clear that X_i may be joint sets, since R is a general relation.

We obtain some rules to find $j-$lower and $j-$upper probabilities in a topologized stochastic approximation spaces for all $j \in \{r, s, p, \gamma, \alpha, \beta\}$.

Definition 4.1.2. Let A be an event in the topologized stochastic approximation space $\mathfrak{S}=(U,R,p,\tau_K)$. Then the $j-$lower (resp. $j-$upper) probability of A for all $j \in \{r, s, p, \gamma, \alpha, \beta\}$ is given by:

$$_j\underline{p}(A)=p\left(\underline{R}_j\, A\right)=p(A^{j\circ}) \text{ (resp.}$$
$$_j\overline{p}(A)=p\left(\overline{R}_j\, A\right)=p(A^{j-})\text{).}$$

Clearly, $0\le {_j\underline{p}}(A)\le 1$ and $0\le {_j\overline{p}}(A)\le 1$ for all $j \in \{r, s, p, \gamma, \alpha, \beta\}$.

Proposition 4.1.1. Let A be an event in the topologized stochastic approximation space $\mathfrak{S}=(U,R,p,\tau_K)$, then

1) $_j\underline{p}(A)\le p(A)\le {_j\overline{p}}(A)$,
2) $_j\underline{p}(\phi)={_j\overline{p}}(\phi)=0$,
3) $_j\underline{p}(U)={_j\overline{p}}(U)=1$,
4) $_j\underline{p}(A^c)=1-{_j\overline{p}}(A)$,
5) $_j\overline{p}(A^c)=1-{_j\underline{p}}(A)$,

for all $j \in \{r, s, p, \gamma, \alpha, \beta\}$.

Proof. By using the properties of $j-$interior and $j-$closure for all $j \in \{r, s, p, \gamma, \alpha, \beta\}$, the proof is obvious. □

In general, properties (6) and (7) in Section 3 do not satisfy in the case of $j-$rough probability for all $j \in \{r, s, p, \gamma, \alpha, \beta\}$. Example 4.1.1 (resp. Example 4.1.2) illustrates that properties (6) and (7) in Section 3 does not satisfy in the case of $p-$rough (resp. $\beta-$rough) probability.

Example 4.1.1. Let $U=\{a,b,c,d\}$ and $R=\{(a,a),(d,d),(a,b)\}$. Then

$$U/R=\{\{d\},\{a,b\}\};$$

Let $K=(U,R)$ be an approximation space and τ_K is the topology associated to K. Thus,

$$\tau_K=\{U,\phi,\{d\},\{a,b\},\{a,b,d\}\}.$$

If $A=\{a\},\ B=\{b\}$, then

$$_p\overline{p}(A\cup B)=\frac{3}{4},$$
$$_p\overline{p}(A)+{_p\overline{p}}(B)-{_p\overline{p}}(A\cap B)=\frac{1}{4}+\frac{1}{4}-0=\frac{2}{4};$$

Thus

$$_p\overline{p}(A\cup B)>{_p\overline{p}}(A)+{_p\overline{p}}(B)-{_p\overline{p}}(A\cap B).$$

Example 4.1.2. Let $K=(U,R)$ be the same approximation space given in Example 4.1.1. If $A=\{b,c,d\},\ B=\{a,c\}$, then

$$_\beta\underline{p}(A\cup B)=1,$$
$$_\beta\underline{p}(A)+{_\beta\underline{p}}(B)-{_\beta\underline{p}}(A\cap B)=\frac{3}{4}+\frac{2}{4}-0=\frac{5}{4};$$

Thus

$$_\beta\underline{p}(A\cup B)<{_\beta\underline{p}}(A)+{_\beta\underline{p}}(B)-{_\beta\underline{p}}(A\cap B).$$

Definition 4.1.3. Let A be an event in the topologized stochastic approximation space

$\mathfrak{S}=(U,R,p,\tau_K)$. Then the $j-$rough probability of A for all $j\in\{r,s,\mathrm{p},\gamma,\alpha,\beta\}$ is denoted by ${}_j p^*(A)$ and is given by

$${}_j p^*(A)=\left\langle {}_j\underline{p}(A),\ {}_j\overline{p}(A)\right\rangle.$$

If A is an $j-$observable event in $\mathfrak{S}$, the $j-$rough probability ${}_j p^*(A)$ for all $j\in\{r,s,\mathrm{p},\gamma,\alpha,\beta\}$ will be the same as the classical probability $p(A)$; that is, ${}_j p^*(A)$ will be reduced to one point.

Moreover,

- If A is externally $j-$unobservable, then ${}_j p^*(A)=\left\langle {}_j\underline{p}(A), 1\right\rangle$.
- If A is internally $j-$unobservable, then ${}_j p^*(A)=\left\langle 0,\ {}_j\overline{p}(A)\right\rangle$.
- If A is totally $j-$unobservable, then ${}_j p^*(A)=\langle 0,\ 1\rangle$.

For all $j\in\{r,s,\mathrm{p},\gamma,\alpha,\beta\}$, the $j-$exact value of the probability of event A is given if it is $j-$observable; if A is roughly $j-$observable, the $j-$lower and the $j-$upper values to the probability of A is given. In the case when the event A is internally (resp. externally) $j-$unobservable, only the $j-$upper (resp. $j-$lower) bound can be determined. But if A is totally $j-$unobservable both the $j-$lower and $j-$upper bounds for the probability of A can not be determined.

Proposition 4.1.2. Let A be an event in the topologized stochastic approximation space $\mathfrak{S}=(U,R,p,\tau_K)$, then

$$\underline{p}(A)\le {}_j\underline{p}(A)\le {}_j\overline{p}(A)\le \overline{p}(A) \text{ for all } j\in\{s,\mathrm{p},\gamma,\alpha,\beta\}.$$

Proof. By using properties of interior, $j-$interior, closure and $j-$closure for all $j\in\{s,\mathrm{p},\gamma,\alpha,\beta\}$, the proof is obvious. □

In general, the above proposition need not be true in the case of $j=r$ as illustrated in the following example.

Example 4.1.3. Let $U=\{a,b,c\}$ and $R=\{(a,a),(a,b),(b,b),(b,c)\}$. Then

$$U/R=\{\{a,b\},\{b,c\}\};$$

Let $K=(U,R)$ be an approximation space and τ_K is the topology associated to K. Thus,

$$\tau_K=\{U,\phi,\{b\},\{a,b\},\{b,c\}\}; \text{ hence,}$$

$$RO(U)=\{U,\phi\}=RC(U).$$

If $A=\{a,c\}$, then

$$\underline{p}(A)=0,\ \overline{p}(A)=\frac{2}{3},\ {}_r\underline{p}(A)=0 \text{ and } {}_r\overline{p}(A)=1.$$

Therefore ${}_r\underline{p}(A)=\underline{p}(A)$ and $\overline{p}(A)\le {}_r\overline{p}(A)$.

We shall use the symbol "$\xrightarrow{\le}$" instead of "$\le$" in the implications between numbers.

Proposition 4.1.3. Let A be an event in the topologized stochastic approximation space $\mathfrak{S}=(U,R,p,\tau_K)$, then the implications between the lower probability and $j-$lower probability of A are given by the following diagram for all $j\in\{s,\mathrm{p},\gamma,\alpha,\beta\}$.

$$\begin{tikzcd}
 & & {}_s\underline{p}(A) \arrow[dr, "\le"] & & \\
\underline{p}(A) \arrow[r, "\le"] & {}_\alpha\underline{p}(A) \arrow[ur, "\le"] \arrow[dr, "\le"'] & & {}_\gamma\underline{p}(A) \arrow[r, "\le"] & {}_\beta\underline{p}(A) \\
 & & {}_\mathrm{p}\underline{p}(A) \arrow[ur, "\le"'] & &
\end{tikzcd}$$

Proof. By Using Proposition 4.1.2 and Proposition 4.1, the proof is obvious.

Proposition 4.1.4. Let A be an event in the topologized stochastic approximation space $\mathfrak{S}=(U,R,p,\tau_K)$, then the implications between the upper probability and $j-$upper probability of A are given by the following diagram for all $j\in\{s,\mathrm{p},\gamma,\alpha,\beta\}$.

$$\begin{tikzcd}
 & & {}_{s}\overline{p}(A) \arrow[dr, "\leq"] & & \\
{}_{\beta}\overline{p}(A) \arrow[r, "\leq"] & {}_{\gamma}\overline{p}(A) \arrow[ur, "\leq"] \arrow[dr, "\leq"] & & {}_{\alpha}\overline{p}(A) \arrow[r, "\leq"] & \overline{p}(A) \\
 & & {}_{\mathrm{p}}\overline{p}(A) \arrow[ur, "\leq"] & &
\end{tikzcd}$$

Proof. By Using Proposition 4.1.2 and Proposition 4.2, the proof is obvious. □

4.2 Near rough distribution function

In this section we shall introduce the concept of near rough (briefly $j-$rough) distribution function of a random variable X for all $j\in\{r,s,\mathrm{p},\gamma,\alpha,\beta\}$. In the following definition we define the $j-$lower and the $j-$upper distribution functions of a random variable X for all $j\in\{r,s,\mathrm{p},\gamma,\alpha,\beta\}$.

Definition 4.2.1. Let X be a random variable in the topologized stochastic approximation space $\mathfrak{S}=(U,R,p,\tau_K)$. Then the $j-$lower distribution (resp. $j-$upper distribution) function of X for all $j\in\{r,s,\mathrm{p},\gamma,\alpha,\beta\}$ is given by:

$$ {}_{j}\underline{F}(x)={}_{j}\underline{p}(X\leq x) \text{ (resp.}$$

$$ {}_{j}\overline{F}(x)={}_{j}\overline{p}(X\leq x)\text{)}.$$

Definition 4.2.2. Let X be a random variable in the topologized stochastic approximation space $\mathfrak{S}=(U,R,p,\tau_K)$. Then the $j-$rough distribution function of X for all $j\in\{r,s,\mathrm{p},\gamma,\alpha,\beta\}$ is denoted by ${}_{j}F^{*}(x)$ and is given by:

$${}_{j}F^{*}(x)=\left\langle {}_{j}\underline{F}(x),\ {}_{j}\overline{F}(x)\right\rangle .$$

Example 4.2.1. Consider the experiment of choosing one card from five cards numbered from one to five. The collection of the five elements forms the outcome space. Hence $U=\{1,2,3,4,5\}$. Let R be a binary relation defined on U such that

$$R=\{(1,1),(1,2),(1,3),(2,3),(3,3),(3,4),(4,4),(4,5),(5,1),(5,2),(5,4),(5,5)\}.$$

Thus $U/R=\{\{1,2,3\},\{3\},\{3,4\},\{4,5\},\{1,2,4,5\}\}$. Let $K=(U,R)$ be an approximation space and τ_K is the topology associated to K. Hence

$$\begin{aligned}
\tau_K=\{&U,\phi,\{3\},\{4\},\{1,2\},\{4,5\},\{3,4\},\{1,2,3\},\\
&\{3,4,5\},\{1,2,4\},\{1,2,3,4\},\{1,2,4,5\}\},\\
\tau_K^{*}=\{&\phi,U,\{1,2,4,5\},\{1,2,3,5\},\{3,4,5\},\{1,2,3\},\{1,2,5\},\{4,5\},\\
&\{1,2\},\{3,5\},\{5\},\{3\}\}.\\
RO(U)=\{&U,\phi,\{3\},\{1,2\},\{4,5\},\{1,2,3\},\{3,4,5\},\{1,2,4,5\}\},\\
RC(U)=\{&\phi,U,\{1,2,4,5\},\{3,4,5\},\{1,2,3\},\{4,5\},\{1,2\},\{3\}\},\\
PO(X)=\{&U,\phi,\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\\
&\{3,4\},\{4,5\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{1,4,5\},\{2,3,4\},\\
&\{2,4,5\},\{3,4,5\},\{1,2,3,4\},\{1,2,4,5\},\{1,3,4,5\},\{2,3,4,5\}\},
\end{aligned}$$

$$PC(X)=\{\phi,U,\{2,3,4,5\},\{1,3,4,5\},\{1,2,4,5\},\{1,2,3,5\},\\ \{3.4.5\},\{2,4,5\},\{2,3,5\},\{1,4,5\},\{1,3,5\},\{1,2,5\},\{1,2,3\},\\ \{4,5\},\{3,5\},\{2,5\},\{2,3\},\{1,5\},\{1,3\},\{1,2\},\{5\},\{3\},\{2\},\{1\}\}.$$

Define the random variable X to be the number on the chosen card. We can construct Table 5.2.1 which contains the $j-$lower and the $j-$upper probabilities of a random variable $X=x$ for $j\in\{r,\mathrm{p}\}$.

Table 4.2.1: $j-$Lower and $j-$upper probabilities of a random variable $X=x$, where $j\in\{r,\mathrm{p}\}$.

X	1	2	3	4	5
${}_r\underline{p}(X=x)$	0	0	$\frac{1}{5}$	0	0
${}_r\overline{p}(X=x)$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$
${}_\mathrm{p}\underline{p}(X=x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	0
${}_\mathrm{p}\overline{p}(X=x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$

Then the $r-$lower and $r-$upper distribution functions of X are

$${}_r\underline{F}(x)=\begin{cases}0 & -\infty<x<3,\\ \frac{1}{5} & 3\le x<\infty.\end{cases} \quad \text{and} \quad {}_r\overline{F}(x)=\begin{cases}0 & -\infty<x<1,\\ \frac{2}{5} & 1\le x<2,\\ \frac{4}{5} & 2\le x<3,\\ 1 & 3\le x<4,\\ \frac{7}{5} & 4\le x<5,\\ \frac{9}{5} & 5\le x<\infty.\end{cases}$$

The $\mathrm{p}-$lower and $\mathrm{p}-$upper distribution functions of X are

$$ {}_{\mathrm{p}}\underline{F}(x)=\begin{cases}0 & -\infty<x<1,\\ \frac{1}{5} & 1\le x<2,\\ \frac{2}{5} & 2\le x<3,\\ \frac{3}{5} & 3\le x<4,\\ \frac{4}{5} & 4\le x<\infty,\end{cases} \quad \text{and} \quad {}_{\mathrm{p}}\overline{F}(x)=\begin{cases}0 & -\infty<x<1,\\ \frac{1}{5} & 1\le x<2,\\ \frac{2}{5} & 2\le x<3,\\ \frac{3}{5} & 3\le x<4,\\ 1 & 4\le x<5,\\ \frac{6}{5} & 5\le x<\infty.\end{cases} $$

4.3 Near rough expectation

In this section we shall introduce the near rough (briefly $j-$rough) expectation of a random variable X for all $j\in\{r,s,\mathrm{p},\gamma,\alpha,\beta\}$. We shall define the $j-$lower and the $j-$upper expectations of a random variable X for all $j\in\{r,s,\mathrm{p},\gamma,\alpha,\beta\}$ in the following definition.

Definition 4.3.1 . Let X be a random variable in the topologized stochastic approximation space $\mathfrak{S}=(U,R,p,\tau_K)$. Then the $j-$lower (resp. $j-$upper) expectation of X for all $j\in\{r,s,\mathrm{p},\gamma,\alpha,\beta\}$ is given by:

$$ {}_j\underline{E}(X)=\sum_{k=1}^{n}x_k \ \ {}_j\underline{p}(X=x_k) $$

$$ \text{(resp. } {}_j\overline{E}(X)=\sum_{k=1}^{n}x_k \ \ {}_j\overline{p}(X=x_k)\text{)}. $$

The $j-$lower (resp. $j-$upper) expectation of *X* is often called the $j-$lower (resp. $j-$upper) mean of *X* and is denoted by ${}_j\underline{\mu}$ (resp. ${}_j\overline{\mu}$) for all $j\in\{r,s,\mathrm{p},\gamma,\alpha,\beta\}$.

Definition 4.3.2. Let X be a random variable in the topologized stochastic approximation space $\mathfrak{S}=(U,R,p,\tau_K)$. Then the $j-$rough expectation of X for all $j\in\{r,s,\mathrm{p},\gamma,\alpha,\beta\}$ is denoted by ${}_jE^*(X)$ and is given by:

$$ {}_jE^*(X)=\left\langle {}_j\underline{E}(X),\ {}_j\overline{E}(X)\right\rangle; $$

The $j-$rough expectation of X also called $j-$rough mean and denoted by ${}_j\mu^*=\left\langle {}_j\underline{\mu},\ {}_j\overline{\mu}\right\rangle$ for all $j\in\{r,s,\mathrm{p},\gamma,\alpha,\beta\}$.

Example 4.3.1. Consider the same experiment as in Example 4.2.1. From Table 4.2.1 it is easy to see the following

- Neither of the $j-$lower and $j-$upper probabilities summed to one for $j\in\{r,p\}$.
- The value 3 of X has $r-$exact probability since $\underline{p}(X)=\overline{p}(X)=\frac{1}{5}$ at $X=3$.
- The values 1,2, and 3 of X has $\mathrm{p}-$exact probabilities since ${}_{\mathrm{p}}\underline{p}(X)={}_{\mathrm{p}}\overline{p}(X)=\frac{1}{5}$ at $X=1,2,3$.

For $j=r$ we get,

The $r-$lower and the $r-$upper expectations of X are

${}_r\underline{\mu}={}_r\underline{E}(X)=0.6$ and ${}_r\overline{\mu}={}_r\overline{E}(X)=5.4$;

The $r-$rough mean (or $r-$rough expectation) of X is ${}_r\mu^*=\langle 0.6,\ 5.4\rangle$;

For $j=\mathrm{p}$ we get,

The $\mathrm{p}-$lower and the $\mathrm{p}-$upper expectations of X are

${}_{\mathrm{p}}\underline{\mu}={}_{\mathrm{p}}\underline{E}(X)=2$ and ${}_{\mathrm{p}}\overline{\mu}={}_{\mathrm{p}}\overline{E}(X)=3.8$;

The $\mathrm{p}-$rough mean (or $\mathrm{p}-$rough expectation) of X is ${}_{\mathrm{p}}\mu^*=\langle 2,\ 3.8\rangle$.

4.4 Near rough variance and near rough standard deviation

In the context of this section we shall introduce the near rough (briefly $j-$rough) variance and the near rough (briefly $j-$rough) standard deviation of a random variable X for all $j \in \{r, s, p, \gamma, \alpha, \beta\}$.

Definition 4.4.1. Let X be a random variable in the topologized stochastic approximation space $\mathfrak{S} = (U, R, p, \tau_K)$. Then the $j-$lower (resp. $j-$upper) variance of X for all $j \in \{r, s, p, \gamma, \alpha, \beta\}$ is given by:

$$_j\underline{V}(X) = {}_j\underline{E}(X - {}_j\underline{\mu})^2 \quad \text{(resp.}$$
$$_j\overline{V}(X) = {}_j\overline{E}(X - {}_j\overline{\mu})^2\text{);}$$

Definition 4.4.2. Let X be a random variable in the topologized stochastic approximation space $\mathfrak{S} = (U, R, p, \tau_K)$. Then the $j-$rough variance of X for all $j \in \{r, s, p, \gamma, \alpha, \beta\}$ is denoted by $_jV^*(X)$ and is given by:

$$_jV^*(X) = \left\langle {}_j\underline{V}(X),\ {}_j\overline{V}(X) \right\rangle.$$

Definition 4.4.3. Let X be a random variable in the topologized stochastic approximation space $\mathfrak{S} = (U, R, p, \tau_K)$. Then the $j-$lower (resp. $j-$upper) standard deviation of X for all $j \in \{r, s, p, \gamma, \alpha, \beta\}$ is given by:

$$_j\underline{\sigma}(X) = \sqrt{{}_j\underline{V}(X)} \quad \text{(resp.}$$
$$_j\overline{\sigma}(X) = \sqrt{{}_j\overline{V}(X)}\text{).}$$

Definition 4.4.4. Let X be a random variable in the topologized stochastic approximation space $\mathfrak{S} = (U, R, p, \tau_K)$. Then the $j-$rough standard deviation of X for all $j \in \{r, s, p, \gamma, \alpha, \beta\}$ is denoted by $_j\sigma^*(X)$ and is given by:

$$_j\sigma^*(X) = \left\langle {}_j\underline{\sigma}(X),\ {}_j\overline{\sigma}(X) \right\rangle.$$

Example 4.4.1. Consider the same experiment as in Example 4.2.1. Then For $j = r$ we get,

$r-$lower and $r-$upper variances of X are

$$_r\underline{V}(X) = 1.152 \text{ and } _r\overline{V}(X) = 14.368,$$

$r-$rough variance of X is $_rV^*(X) = \langle 1.152, 14.368 \rangle$, and

$r-$rough standard deviation of X is $_r\sigma^*(X) = \langle 1.073, 3.791 \rangle$.

For $j = p$ we get,

$p-$lower and $p-$upper variances of X are

$$_p\underline{V}(X) = 1.2 \text{ and } _p\overline{V}(X) = 2.648;$$

$p-$rough variance of X is $_pV^*(X) = \langle 1.2, 2.648 \rangle$.

Finally, the $p-$rough standard deviation of X is $_p\sigma^*(X) = \langle 1.095, 1.627 \rangle$.

5. Conclusions

In this paper, we used topological concepts to introduce definitions to near rough probability, near rough distribution function, near rough expectation,..etc. We generalize near rough probability in the frameworks of topological spaces. We believe such generalization will be useful in digital topology as well as biomathematics. Our approach is to topologize information systems. We connect probability and topological spaces.

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