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Near rough cluster points in topologized approximation spaces

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Abstract: *The classes of near open sets can be considered as rich sources for elementary concepts in information systems. These classes have been used extensively in abstract topological results. The purpose of this paper is to spotlight on using these classes as tools for finding near rough cluster points. Basic notions of near rough cluster points are introduced. The topology induced by binary relations is used to generalize the basic near rough concepts. Moreover, proved results, examples and counter examples are provided. The topological structure which suggested in this paper opens up the way for applying rich amount of topological facts and methods in the process of granular computing.*

Keywords: *Topological space; Topologized approximation space; Near rough cluster points*

1. Introduction

One of the most powerful notions in system analysis is the concept of topological structures [1] and their generalizations. Many works have appeared recently for example in structural analysis [2], in chemistry [3], and physics [4]. The purpose of the present work is to put a starting point for the applications of abstract topological theory into rough set analysis. Rough set theory, introduced by Pawlak in 1982 [5], is a mathematical tool that supports also the uncertainty reasoning but qualitatively. In this paper, we shall integrate some ideas in terms of concepts in topology. Topology is a branch of mathematics, whose concepts exist not only in almost all branches of mathematics, but also in many real life applications.

2. Preliminaries

This section presents a review of some fundamental notions of topological spaces and rough set theory.

A topological space [1] is a pair (X,τ) consisting of a set X and a family τ of subsets of X satisfying the following conditions:

(T1) $\phi \in \tau$ and $X \in \tau$.

(T2) τ is closed under arbitrary union.

(T3) τ is closed under finite intersection.

Throughout this paper (X,τ) denotes a topological space, the elements of X are called points of the space, the subsets of X belonging to τ are called open sets in the space, the complement of the subsets of X belonging to τ are called closed sets in the space, and the family of all $\tau-$closed subsets of X is denoted by τ^*. The family τ of open subsets of X is also called a topology for X.

A family $\mathrm{B} \subseteq \tau$ is called a base for (X,τ) iff every nonempty open subset of X can be represented as a union of subfamily of B. Clearly, a topological space can have many bases. A family $\mathrm{S} \subseteq \tau$ is called a subbase iff the family of all finite intersections of S is a base for (X,τ).

The τ-closure of a subset $A \subseteq X$ is denoted by A^- and is given by $A^- = \cap\{F \subseteq X : A \subseteq F \text{ and } F \in \tau^*\}$. Evidently, A^- is the smallest closed subset of X which contains A. Note that A is closed iff $A = A^-$.

The τ-interior of a subset $A \subseteq X$ is denoted by A° and is given by $A^\circ = \cup\{G \subseteq X : G \subseteq A \text{ and } G \in \tau\}$. Evidently, A° is the union of all open subsets of X which contained in A. Note that A is open iff $A = A^\circ$.

Motivation for rough set theory has come from the need to represent subsets of a universe in terms of equivalence classes of a partition of that universe. The partition characterizes a topological space, called approximation space $K = (X,R)$, where X is a set called the universe and R is an equivalence relation [6, 7]. The equivalence classes of R are also known as the granules, elementary sets or blocks; we shall use $R_x \subseteq X$ to denote the equivalence class containing $x \in X$, and X/R to denote the set of all elementary sets of R. In the approximation space, we consider two operators, the upper and lower approximations of subsets: Let $A \subseteq X$, then the lower approximation (resp. the upper approximation) of A is given by

$$\underline{R}A = \{x \in X : R_x \subseteq A\}$$
$$\left(\text{resp. } \overline{R}A = \{x \in X : R_x \cap A \neq \phi\}\right).$$

Pawlak noted that [7] the approximation space $K=(X,R)$ with equivalence relation R defines a uniquely topological space (X,τ_K) where τ_K is the family of all clopen sets and X/R is a base of τ_K. Moreover the lower (resp. upper) approximation of any subset $A\subseteq X$ is exactly the interior (resp. closure) of the subset A. In this section we shall generalize Pawlak's concepts in the case of general relations. Hence the approximation space $K=(X,R)$ with general relation R defines a uniquely topological space (X,τ_K) where τ_K is the topology associated to K (i.e. τ_K is the family of all open sets in (X,τ_K) and $X/R=\{xR : x\in X\}$ is a subbase of τ_K, where $xR=\{y\in X : xRy\}$). We shall give this hypothesis in the following definition.

Definition 2.1. Let $K=(X,R)$ be an approximation space with general relation R and τ_K is the topology associated to K. Then the triple $\kappa=(X,R,\tau_K)$ is called a topologized approximation space.

The following definition introduces lower and upper approximations in a topologized approximation space $\kappa=(X,R,\tau_K)$.

Definition 2.2. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space. If $A\subseteq X$, then the lower approximation (resp. upper approximation) of A is defined by

$$\underline{R}A=A^{\circ} \text{ (resp. } \overline{R}A=A^{-}\text{)}.$$

3. Near rough cluster points in topologized approximation spaces

The present section is devoted to introduce the near rough cluster points by applying the concepts of near open sets. We study approximation spaces from topological view and obtain some rules to find lower and upper approximations in several ways in approximation spaces with general relations. We shall recall some definitions about some classes of near open sets which are essential for our present study. Some forms of near open sets are introduced in the following definition.

Definition 3.1. Let (X,τ) be a topological space, then the subset $A\subseteq X$ is called:

i) Regular open [8] (briefly $r-$open) if $A=A^{-\circ}$.

ii) Semi-open [9] (briefly $s-$open) if $A\subseteq A^{\circ -}$.

ii) Pre-open [10] (briefly $p-$open) if $A\subseteq A^{-\circ}$.

iv) γ -open [11] ($b-$open [12]) if $A\subseteq A^{\circ -}\cup A^{-\circ}$.

v) α -open [13] if $A\subseteq A^{\circ - \circ}$.

vi) β -open [14] (semi-pre-open [15]) if $A\subseteq A^{-\circ -}$.

Notice 3.1.
i) The complement of an $r-$open (resp. $s-$open, $p-$open, γ-open, α-open and β-open) set is called $r-$closed (resp. $s-$closed, $p-$closed , γ-closed, α-closed and β-closed) set .
ii) The family of all $r-$open (resp. $s-$open, $p-$open, γ-open, α-open and β-open) sets of (X,τ) is denoted by $RO(X)$ (resp. $SO(X)$, $PO(X)$, $\gamma O(X)$, $\alpha O(X)$ and $\beta O(X)$).
iii) The family of all $r-$closed (resp. $s-$closed, $p-$closed, γ- closed, α- closed and β-closed) sets of (X,τ) is denoted by $RC(X)$ (resp. $SC(X)$, $PC(X)$, $\gamma C(X)$, $\alpha C(X)$ and $\beta C(X)$).

The aim of the following example is to illustrate the existence of spaces in which the above classes of near open sets and near closed sets are not coincided and do not have the discrete structure.

Example 3.1. Let $X=\{a,b,c,d\}$ and $\tau=\{X,\phi,\{a,b\},\{d\},\{a,b,d\}\}$. Then the classes of near open sets are

$$RO(X)=\{X,\phi,\{d\},\{a,b\}\}$$
$$SO(X)=\{X,\phi,\{d\},\{a,b\},\{c,d\},\{a,b,c\},\{a,b,d\}\}$$
$$PO(X)=\{X,\phi,\{a\},\{b\},\{d\},\{a,b\},\{a,d\},\{b,d\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}$$
$$\gamma O(X)=\{X,\phi,\{a\},\{b\},\{d\},\{a,b\},\{a,d\},\{b,d\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}$$

$$\alpha O(X)=\{X,\phi,\{d\},\{a,b\},\{a,b,d\}\}$$
$$\beta O(X)=\{X,\phi,\{a\},\{b\},\{d\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}.$$

The following definition is given to introduce the near interior of a subset A of X in a topological space (X,τ).

Definition 3.2. Let (X,τ) be a topological space and $A\subseteq X$, then the near interior (briefly $j-$interior) of A is denoted by $A^{j\circ}$ for all $j\in\{r,s,p,\gamma,\alpha,\beta\}$ and is defined by

$$A^{j\circ}=\cup\{G\subseteq X : G\subseteq A,\ G \text{ is a } j-\text{open set}\}.$$

The aim of the following definition is to introduce the near closure of a subset A of X in a topological space (X,τ).

Definition 3.3. Let (X,τ) be a topological space and $A\subseteq X$, then the near closure (briefly $j-$closure) of A is denoted by A^{j-} for all $j\in\{r,s,p,\gamma,\alpha,\beta\}$ and is defined by

$$A^{j-}=\cap\{H\subseteq X : A\subseteq H,\ H \text{ is a } j-\text{closed set}\}$$

Near lower and near upper approximations in a topologized approximation space $\kappa=(X,R,\tau_K)$ are introduced in the following two definitions.

Definition 3.4. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space. If $A\subseteq X$, then the near lower approximation (briefly $j-$lower approximation) of A is denoted by $\underline{R}_j\,A$ and is defined by

$$\underline{R}_j\,A=A^{j\circ},\ \text{where } j\in\{r,p,s,\gamma,\alpha,\beta\}.$$

Definition 3.5. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space. If $A\subseteq X$, then the near upper approximation (briefly $j-$upper approximation) of A is denoted by $\overline{R}_j\,A$ and is defined by

$$\overline{R}_j\,A=A^{j-},\ \text{where } j\in\{r,p,s,\gamma,\alpha,\beta\}.$$

Next, we introduce the definition of rough cluster points of a subset A of X in a topologized approximation space $\kappa=(X,R,\tau_K)$.

Definition 3.6. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space. The point $p\in X$ is said to be a rough cluster point of a subset A of X if

$$\forall B\subseteq X \text{ such that } p\in\underline{R}B, \text{ then } (\underline{R}B-\{p\})\cap A\neq\phi.$$

The set of all rough cluster points of A is denoted by $R'A$ and is called the rough derived set of A.

Theorem 3.1. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space. Then a subset A of X is a τ_K-closed if and only if $R'A\subseteq A$.

Proof.

Suppose that A is a τ_K-closed subset of X and let $p\notin A$ (i.e. $p\in X-A$). Then $X-A$ is a τ_K-open set. Thus

$$p\in\underline{R}(X-A)=(X-A)^{\circ}=(X-A) \text{ and } \underline{R}(X-A)\cap A=\phi.$$

Hence $p\notin R'A$. Therefore $R'A\subseteq A$.

Conversely, let $R'A\subseteq A$, need to show that A is a τ_K-closed subset of X. Now, let $p\in X-A$ then $p\notin R'A$, hence there exists a subset $B\subseteq X$ such that $p\in\underline{R}B$ and $(\underline{R}B-\{p\})\cap A=\phi$. But $p\notin A$, hence $\underline{R}B\cap A=\phi$. So $p\in\underline{R}B\subseteq X-A$ and

$$X-A=\bigcup_{p\in X-A}\{p\}\subseteq\cup\,\underline{R}B=\cup\,B^{\circ}\subseteq X-A.$$

Thus $X-A$ is a union of τ_K-open sets, which is τ_K-open. Hence A is a τ_K-closed subset of X. □

Example 3.2. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space such that $X=\{a,b,c,d\}$, $X/R=\{\{d\},\{a,b\}\}$. Then $\mathrm{S}=\{\phi,\{d\},\{a,b\}\}$, $\mathrm{B}=\{X,\phi,\{d\},\{a,b\}\}$, and

$\tau=\{X,\phi,\{d\},\{a,b\},\{a,b,d\}\}$,

$\tau^{*}=\{\phi,X,\{a,b,c\},\{c,d\},\{c\}\}$.

If $A=\{a,b,c\}$, then $R'A=\{a,b,c\}$. Thus $R'A\subseteq A$ and A is τ_K-closed subset of X.

The following definition introduces the definition of near rough (briefly $j-$rough) cluster points of a subset A of X in a topologized approximation space $\kappa=(X,R,\tau_K)$ for all $j\in\{r,p,s,\gamma,\alpha,\beta\}$.

Definition 3.7. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space. The point $p\in X$ is said to be a $j-$rough cluster point of a subset A of X for all $j\in\{r,p,s,\gamma,\alpha,\beta\}$, if

$$\forall B\subseteq X \text{ such that } p\in\underline{R}_j\,B, \text{ then } (\underline{R}_j\,B-\{p\})\cap A\neq\phi.$$

The set of all $j-$rough cluster points of A is denoted by $R'_j\,A$ and is called the $j-$rough derived set of A for all $j\in\{r,p,s,\gamma,\alpha,\beta\}$.

Theorem 3.2. Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space. Then a subset A of X is a $j-$closed for all $j\in\{p,s,\gamma,\alpha,\beta\}$ if and only if $R'_j A\subseteq A$.

Proof.

We shall prove this theorem in the case of $j=\beta$ and the other cases can be proved similarly. Suppose that A is a $\beta-$closed subset of X and let $p\notin A$ (i.e. $p\in X-A$). Then $X-A\in\beta O(X)$. Thus

$$p\in \underline{R}_\beta(X-A)=(X-A)^{\beta\circ}=(X-A) \text{ and } \underline{R}_\beta(X-A)\cap A=\phi.$$

Hence $p\notin R'_\beta A$. Therefore $R'_\beta A\subseteq A$.

Conversely, let $R'_\beta A\subseteq A$; need to show that A is a $\beta-$closed subset of X. Now, let $p\in X-A$ then $p\notin R'_\beta A$; hence there exists a subset $B\subseteq X$ such that $p\in\underline{R}_\beta B$ and $(\underline{R}_\beta B-\{p\})\cap A=\phi$. But $p\notin A$, hence $\underline{R}_\beta B\cap A=\phi$. So $p\in\underline{R}_\beta B\subseteq X-A$ and

$$X-A=\bigcup_{p\in X-A}\{p\}\subseteq\cup\,\underline{R}_\beta B=\cup\, B^{\beta\circ}\subseteq X-A\cdot$$

Thus $X-A$ is a union of $\beta-$open sets, which is $\beta-$open. Hence A is a $\beta-$closed subset of X. □

Example 3.3. Let $\kappa=(X,R,\tau_K)$ be the topologized approximation space which is given in Example 3.2 and $A=\{a,b\}$. Then $R'_s A=\{a,b\}$, thus $R'_s A\subseteq A$ and A is $s-$closed subset of X.

In general Theorem 3.2 can not be satisfied in the case of $j=r$, as the following example illustrates.

Example 3.4. Let $\kappa=(X,R,\tau_K)$ be the topologized approximation space which is given in Example 3.2 and $A=\{c\}$. Then $R'_r A=\{c\}$. Thus $R'_r A\subseteq A$; But A is not an $r-$closed subset of X, since

$$RC(X)=\{X,\phi,\{a,b,c\},\{c,d\}\}.$$

Theorem 3.3. Let A be a subset of X in the topologized approximation space $\kappa=(X,R,\tau_K)$. Then $x\in\overline{R}A$ if and only if $\underline{R}B\cap A\neq\phi$ for each $B\subseteq X$ and $x\in\underline{R}B$.

Proof.

Let $x\in\overline{R}A$ and $x\in\underline{R}B$ for some $B\subseteq X$. Assume $\underline{R}B\cap A=\phi$. This implies that $A\subseteq X-\underline{R}B$, but $X-\underline{R}B=X-B^\circ$ which is a τ_K-closed set. Hence $x\in X-\underline{R}B$, since $x\in\overline{R}A$ and this leads to a contradiction. Therefore $\underline{R}B\cap A\neq\phi$.

Conversely, suppose that for each $B\subseteq X$, $x\in\underline{R}B$ then $\underline{R}B\cap A\neq\phi$. Let $x\notin\overline{R}A$, but $\overline{R}A=A^-$ which is τ_K-closed, then there exists a closed set $F\subseteq X$ such that $F\supseteq A$ and $x\notin F$. Hence $X-F$ is τ_K-open set containing x. Thus $x\in\underline{R}(X-F)=(X-F)^\circ=X-F$ and $\underline{R}(X-F)\cap A=\phi$; that is there exists a subset $B=X-F$ of X such that $\underline{R}B\cap A=\phi$, which leads to a contradiction. Therefore $x\in\overline{R}A$. □

Theorem 3.4. Let A be a subset of X in the topologized approximation space $\kappa=(X,R,\tau_K)$. Then $x\in\overline{R}_j A$ for all $j\in\{p,s,\gamma,\alpha,\beta\}$ if and only if $\underline{R}_j B\cap A\neq\phi$ for each $B\subseteq X$ and $x\in\underline{R}_j B$.

Proof.

We shall prove this theorem in the case of $j=\beta$ and the other cases can be proved similarly.

Let $x\in\overline{R}_\beta A$ and $x\in\underline{R}_\beta B$ for some $B\subseteq X$. Assume $\underline{R}_\beta B\cap A=\phi$. This implies that $A\subseteq X-\underline{R}_\beta B$, but

$X-\underline{R}_\beta B=X-B^{\beta\circ}$ which is a $\beta-$closed set.

Hence $x\in X-\underline{R}_\beta B$, since $x\in\overline{R}_\beta A$ and this leads to a contradiction. Therefore $\underline{R}_\beta B\cap A\neq\phi$.

Conversely, suppose that for each $B\subseteq X$, $x\in\underline{R}_\beta B$ then $\underline{R}_\beta B\cap A\neq\phi$. Let $x\notin\overline{R}_\beta A$, but $\overline{R}_\beta A=A^{\beta-}$ which is $\beta-$closed, then there exists a $\beta-$closed set $F\subseteq X$ such that $F\supseteq A$ and $x\notin F$. Hence $X-F$ is $\beta-$open set containing x, thus

$$x\in\underline{R}_\beta(X-F)=(X-F)^{\beta\circ}=X-F \text{ and } \underline{R}_\beta(X-F)\cap A=\phi,$$

that is there exists a subset $B = X - F$ of X such that $\underline{R}_{\beta} B \cap A = \phi$, which leads to a contradiction. Therefore $x \in \overline{R}_{\beta} A$. □

Theorem 3.5. Let A be a subset of X in the topologized approximation space $\kappa = (X, R, \tau_K)$. Then $\overline{R} A = A \cup R' A$.

Proof.

By Theorem 3.1, we get $R' A \subseteq \overline{R} A$, then

$$A \cup R' A \subseteq A \cup \overline{R} A = \overline{R} A.$$

For the converse inclusion, let $x \in \overline{R} A$, then either $x \in A$ and hence $x \in A \cup R' A$, or $x \notin A$, and hence by Theorem 3.3, for each $B \subseteq X$, $x \in \underline{R} B$, we get $\underline{R} B \cap A \neq \phi$, then $x \in R' A$ and hence $x \in A \cup R' A$. Thus $\overline{R} A \subseteq A \cup R' A$. Therefore $\overline{R} A = A \cup R' A$. □

Theorem 3.6. Let A be a subset of X in the topologized approximation space $\kappa = (X, R, \tau_K)$. Then $\overline{R}_j A = A \cup R'_j A$ for all $j \in \{p, s, \gamma, \alpha, \beta\}$.

Proof.

We shall prove this theorem in the case of $j = \alpha$ and the other cases can be proved similarly. By Theorem 3.2, we get $R'_{\alpha} A \subseteq \overline{R}_{\alpha} A$, then

$$A \cup R'_{\alpha} A \subseteq A \cup \overline{R}_{\alpha} A = \overline{R}_{\alpha} A.$$

For the converse inclusion, let $x \in \overline{R}_{\alpha} A$, then either $x \in A$ and hence $x \in A \cup R'_{\alpha} A$, or $x \notin A$, and hence by Theorem 3.4, for each $B \subseteq X$, $x \in \underline{R}_{\alpha} B$, we get $\underline{R}_{\alpha} B \cap A \neq \phi$, then $x \in R'_{\alpha} A$ and hence $x \in A \cup R'_{\alpha} A$. Thus $\overline{R}_{\alpha} A \subseteq A \cup R'_{\alpha} A$. Therefore $\overline{R}_{\alpha} A = A \cup R'_{\alpha} A$. □

4. Conclusions

In this paper, we used topological concepts to introduce definitions to near rough cluster points. We generalize near rough cluster points in the frameworks of topologized approximation spaces. We believe such generalization will be useful in digital topology as well as biomathematics.

References

[1] J. Kelley, General topology, Van Nostrand Company, 1955.

[2] C. Largeron, S. Bonnevay, A Pretopological Approach for Structural Analysis, Information Sci. 144 (2002) 185–196.

[3] B.M.R. Stadler, P.F. Stadler, Generalized topological spaces in evolutionary theory and combina-torial chemistry, J. Chem. Inf. Comput. Sci. 42 (2002) 577–585.

[4] A. Galton, A generalized topological view of motion in discrete space, Theor. Comput. Sci. (2002).

[5] Z. Pawlak, Rough sets, Int. J. Information Comput. Sci. 11 (5) (1982) 341–356.

[6] T.Y. Lin, Topological and fuzzy rough sets, in: R. Slowinski (Ed.), Decision Support by Experience—Application of the Rough Sets Theory, Kluwer Academic Publishers, 1992, pp. 287–304.

[7] Z. Pawlak, Rough Sets, Theoretical Aspects of Reasoning about Data, Kluwer Academic, Boston, 1991.

[8] M. Stone, Application of the theory of Boolian rings to general topology, Trans. Amer, Math. Soc., 41(1937), 374-481.

[9] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer Math. Monthly, 70(1963), 36-41.

[10] A. S. Mashhour, M. E. Abd El-Monsef, S. N. El-Deeb, On pre continuous and weak pre continuous mappings, Proc. Math. Phys. Soc. Egypt, 53(1982), 47-53.

[11] A. A. El-Atik, A study of some types of mappings on topological spaces, M.Sc. thesis, Tanta Univ. (1997).

[12] D. Andrijević, On b-open sets, Mat. Vesnik, 48(1996), 59-64.

[13] O. Njastad, On some classes of nearly open sets, Pacific J. Math. 15(1965), 961-970.

[14] M. E. Abd El-Monsef, S. N. El-Deeb, R. A. Mahmoud, β-open sets, β-continuous mappings, Bull. Fac. Sc. Assuit Univ., 12(1983), 77-90.

[15] D. Andrijević, Semi-preopen sets, ibid. 38(1986), 24-32.

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