On Weakly Primary Submodules

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Abstract: *Let R be a commutative ring with non-zero identity. A proper submodule N of an R-module M is weakly primary if $0 \neq rm \in N(r \in R, m \in M)$ implies $m \in N$ or $r^n M \subseteq N$ for some positive integer n. A number of results concerning weakly primary submodules are given. For example, we give other characterization of weakly primary submodules. Also various properties of weakly primary submodules are considered. For example we show that if N is a weakly primary submodule of an R-module M which is not primary, then $IN = 0$ for every ideal $I \subseteq \sqrt{(N:M)}$.*

1. Introduction

Weakly primary submodules in a commutative ring with non-zero identity have been introduced by S. E. Atani and F. Farzalipour in [1]. In [2], the concept of weakly prime submodules was introduced and a number of results concerning weakly prime submodules were given. The weakly primary submodules and the weakly prime submodules are different concepts. These two concepts are extensions of the concepts of weakly primary and weakly prime ideals that have the following definitions: A proper ideal P of R is said to be weakly prime ideal if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$, and it is weakly primary ideal if $0 \neq ab \in P$ implies $a \in P$ or $b^n \in P$ for some positive integer n. Several authors have studied in details weakly primary and weakly prime ideals and prove many important results about these two concepts, see [1],[3],[4],[5].

In this paper, we prove several results concerning the weakly primary submodules of a commutative ring. Some of our results are analogous to the results given in [2]. The corresponding results are obtained by modification. We also give various properties of weakly primary submodules.

Before we state some results and properties concerning weakly primary submodules, let us introduce some notations.

Throughout this paper all rings will be commutative with non-zero identity. A proper submodule N of an R-module M is said to be weakly prime if $0 \neq rm \in N (r \in R, \ m \in M)$ implies $m \in N$ or $rM \subseteq N$ [2]. If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R : rM \subseteq N\}$ will be denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M.

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It is known that if N is a primary submodule of an R-module M, then $(N : M)$ is

a primary ideal of R and $\sqrt{(N : M)} = \{r \in R : r^n M \subseteq N$ for some positive integer $n\}$ is a prime ideal of R, [6]. In Section 2, we show that this is not in general true in the case of a weakly primary submodule. We investigate the conditions that make these results satisfied in the case of weakly primary submodules.

In [7] some results were proved concerning weakly primary submodules that are not primary. In Section 3, we prove that if N is a weakly primary submodule of an R-module M that is not primary, then $IN = 0$ for every ideal $I \subseteq \sqrt{(N : M)}$. We also give a characterization of weakly primary submodules. We show that a submodule N of an R-module M is weakly primary if and only if for any ideal I of R and any submodule D of M with $0 \neq ID \subseteq N$, either $I \subseteq \sqrt{(N : M)}$ or $D \subseteq N$.

Finally, in Section 4, we show that the direct sum of a finite number of submodules that contain only one weakly primary submodule is weakly primary, however the direct sum of weakly primary submodules is not in general weakly primary.

2. Weakly Primary Submodules

Recall that a proper submodule N of a module M over a commutative ring R is said to be weakly primary if whenever $0 \neq rm \in N (r \in R, m \in M)$, then $m \in N$ or $r^n M \subseteq N$ for some positive integer *n*.

Remarks 2.1

1) Clearly, every primary submodule of a module is weakly primary submodule. However, since 0 is always weakly primary (by the definition), a weakly primary submodule need not be primary. Thus the weakly primary submodule concept is a generalization of the concept of primary submodule.
2) We can show directly from the definitions that every weakly prime submodule is weakly primary, however the converse is not in general true since for example if $R = \mathbb{Z}$, the set of integers, $M = \mathbb{Z} \times \mathbb{Z}$ and $L = (4,0)\mathbb{Z} + (0,1)\mathbb{Z}$, then L is a weakly primary submodule [8], however it is not weakly prime submodule since $0 \neq 2(2,1) \in L$. But neither $2M \subseteq L$ nor $(2,1) \in L$.
3) It is known that if N is a primary submodule of an R-module M, then $(N:M)$ is a primary ideal of R and $\sqrt{(N:M)}$ is a prime ideal of R, [6]. Contrary to what happens for a primary submodules, if N is a weakly primary submodule, the ideal $(N:M)$ is not in general a weakly primary ideal of R, and also $\sqrt{(N:M)}$ is not in general a weakly prime ideal of R. For example: Let M denotes the cyclic $\mathbb{Z}$–module $\mathbb{Z}/6\mathbb{Z}$. Take $N = \{0\}$. Certainly N is a weakly primary submodule of M, but neither $(N:M) = 6\mathbb{Z}$ is a weakly primary ideal of R nor $\sqrt{(N:M)} = 6\mathbb{Z}$ is a weakly prime ideal of R. However we have the following results:

Proposition 2.2 Let R be a commutative ring, M a faithful cyclic R-module and N is a weakly primary submodule of M. Then $(N:M)$ is a weakly primary ideal of R.

Proof. Assume $M = Rx$ and let $0 \neq ab \in (N:M)$ with $a \notin (N:M)$, then there exists $r \in R$ such that $a(rx) \notin N$, so $ax \notin N$. As $0 \neq abM \subseteq N$, it follows that $0 \neq abx \in N$ (for if $abx = 0$, then $ab \in (0:x) = (0:M) = 0$, a contradiction). Since N is a weakly primary submodule of M, $b^n \in (N:M)$ for some positive integer n. Thus $(N:M)$ is a weakly primary ideal of R. □

Proposition 2.3 Let R be a commutative ring, M a faithful cyclic R-module and N is a weakly primary submodule of M. Then $\sqrt{(N:M)}$ is a weakly prime ideal of R.

Proof. Assume $M = Rx$ and let $0 \neq ab \in \sqrt{(N:M)}$ with $a \notin \sqrt{(N:M)}$, then $a \notin (N:M)$. As in the proof of the previous proposition, $b^n \in (N:M)$ for some positive integer n. Thus $b \in \sqrt{(N:M)}$ and $\sqrt{(N:M)}$ is a weakly prime ideal of R. □

Proposition 2.4 Let R be a commutative ring, M a P-primary R-module and N is a weakly primary submodule of M. Then $(N:M)$ is a weakly primary ideal of R.

Proof. Let $0 \neq ab \in (N:M)$ with $a \notin (N:M)$, then there exists $m \in M - N$ such that $am \notin N$. As $0 \neq abM \subseteq N$, we have $abm \in N$. If $abm = 0$, then $ab \in (0:m) = (0:M) = P$, so $b^n \in P \subseteq (N:M)$ for some positive integer n. If $abm \neq 0$, since N is a weakly primary submodule of M, $b^n \in (N:M)$ for some positive integer n. Thus $(N:M)$ is a weakly primary ideal of R. □

Proposition 2.5 Let R be a commutative ring, M a P-primary R-module and N is a weakly primary submodule of M. Then $\sqrt{(N:M)}$ is a weakly prime ideal of R.

Proof. Let $0 \neq ab \in \sqrt{(N:M)}$ with $a \notin \sqrt{(N:M)}$, then $a \notin (N:M)$. Thus there exists $m \in M - N$ such that $am \notin N$. As in the proof of the previous proposition, $b^n \in (N:M)$ for some positive integer n. Thus $b \in \sqrt{(N:M)}$ and $\sqrt{(N:M)}$ is a weakly prime ideal of R. □

3. Characterization of Weakly Primary Submodules

The following two results concerning weakly primary submodules that are not
primary were proved in [7].

Proposition 3.1 Let R be a commutative ring, M an R-module, and N a weakly primary submodule of M. If N is not primary submodule of M, then $(N:M)N = 0$.

Proposition 3.2 Let R be a commutative ring, M a multiplication R-module, and N a weakly primary submodule of M. If N is not primary submodule of M, then $N^2 = 0$.

We give now a generalization of Proposition 3.1 which is the corollary of the next theorem.

Theorem 3.3 Let N be a weakly primary submodule of an R-module M. If $\sqrt{(N:M)}\, N \neq 0$, then N is a primary submodule of M.

Proof. Let $rm \in N$ for some $r \in R$ and $m \in M$. If $rm \neq 0$, then N is weakly primary gives $m \in N$ or $r^n \in (N:M)$ for some positive integer n. So assume that $rm = 0$. First suppose that $rN \neq 0$, then there exists $s \in N$ such that $rs \neq 0$. Thus $0 \neq rs = r(m+s) \in N$, so N is weakly primary gives $r^n \in (N:M)$ for some positive integer n, or $(m+s) \in N$. Hence $r^n \in (N:M)$ for some positive integer n, or $m \in N$. So we can assume that $rN = 0$. Next suppose that $m\sqrt{(N:M)} \neq 0$, say $mk \neq 0$ where $k \in \sqrt{(N:M)}$. Then $0 \neq mk = (r+k)m \in N$. Thus $(r+k) \in \sqrt{(N:M)}$ or $m \in N$. So

$r \in \sqrt{(N:M)}$ or $m \in N$. Therefore $r^n \in (N:M)$ for some positive integer *n*, or $m \in N$. So we can assume $m\sqrt{(N:M)} = 0$. Since $\sqrt{(N:M)}\ N \neq 0$, there exists $t \in \sqrt{(N:M)}$ and $d \in N$ with $td \neq 0$. Then $0 \neq td = (r+t)(m+d) \in N$, so $(r+t) \in \sqrt{(N:M)}$ or $(m+d) \in N$. Thus $r \in \sqrt{(N:M)}$ or $m \in N$. Therefore $r^n \in (N:M)$ for some positive integer *n*, or $m \in N$. So N is a primary submodule of M. □

Now, the following result follows immediately from Theorem 3.3.

Corollary 3.4 Let R be a commutative ring, M an R-module and N a weakly primary submodule of M. If N is not primary submodule of M, then for any ideal I of R such that $I \subseteq \sqrt{(N:M)}$, then $IN = 0$. In particular, $\sqrt{(N:M)}\,N = 0$.

Remark 3.5 We can give another proof the result stated in Proposition 3.2 by using Corollary 3.4 as follows: Since M is a multiplication module, $N = (N:M)M$. Therefore $N^2 = (N:M)^2 M = (N:M)N \subseteq \sqrt{(N:M)}N$. Since N is weakly primary, but not primary, $\sqrt{(N:M)}\ N = 0$. Thus $N^2 = 0$.

We next give a characterization of weakly primary submodules.

Theorem 3.6 Let R be a commutative ring, M an R-module, and N a proper submodule of M. Then N is a weakly primary submodule of M if and only if for every ideal I of R and every submodule D of M with $0 \neq ID \subseteq N$, either $I \subseteq \sqrt{(N:M)}$ or $D \subseteq N$.

Proof. $\Leftarrow$) Suppose that $0 \neq sm \in N$ where $s \in R$ and $m \in M$. Take $I = Rs$ and $D = Rm$. Then $0 \neq ID \subseteq N$, so either $I \subseteq \sqrt{(N:M)}$ or $D \subseteq N$; hence either $s \in \sqrt{(N:M)}$ or $m \in N$.

Thus either $s^n M \subseteq N$ for some positive integer n or $m \in N$. Therefore N is a weakly primary submodule of M.

$\Rightarrow$) Suppose that N is a weakly primary submodule of M. If N is primary, then the result is clear. So we can assume that N is a weakly primary submodule of M that is not primary. Let $0 \neq ID \subseteq N$ with $x \in D - N$. We show that $I \subseteq \sqrt{(N:M)}$.

Let $r \in I$. If $0 \neq rx$, then N is weakly primary gives $r \in \sqrt{(N:M)}$. So assume that $0 = rx$. First suppose that $rD \neq 0$, say $rd \neq 0$ where $d \in D$. If $d \notin N$, then $r \in \sqrt{(N:M)}$. If $d \in N$, then $r(d+x) = rd \in N$, so $r \in \sqrt{(N:M)}$ or $(d+x) \in N$. Thus $r \in \sqrt{(N:M)}$. So we can assume that $rD = 0$. Suppose that $Ix \neq 0$, say $ax \neq 0$ where $a \in I$. Then N is weakly primary gives $a \in \sqrt{(N:M)}$. As $(r+a)x = ax \in N$, we get $r \in \sqrt{(N:M)}$. Therefore, we can assume that $Ix = 0$. Since $ID \neq 0$, there exist $b \in I$ and $d_1 \in D$ such that $bd_1 \neq 0$. As $\sqrt{(N:M)}\, N = 0$ (by Corollary 3.4) and $0 \neq b(d_1 + x) = bd_1 \in N$ we can divide the proof into the following cases:

Case1. $b \in \sqrt{(N:M)}$ and $(d_1 + x) \notin N$.

Since $0 \neq (r+b)(d_1+x) = bd_1 \in N$, we obtain $r + b \in \sqrt{(N:M)}$, so

$r \in \sqrt{(N:M)}$.

Case2. $b \notin \sqrt{(N:M)}$ and $(d_1 + x) \in N$.

As $0 \neq bd_1 \in N$. We have $d_1 \in N$, so $x \in N$ which is a contradiction.

Therefore in all the cases, we have $r \in \sqrt{(N:M)}$. Thus $I \subseteq \sqrt{(N:M)}$ and the proof is complete. □

Proposition 3.7 Let M be a multiplication R-module and N_1 , N_2 , ..., N_k be submodules of M. Let N be a primary submodule of M with $\sqrt{(N:M)}\, M = 0$. If $\prod_{i=1}^{k} N_i \subseteq N$, then $\prod_{i=1}^{k} N_i = \bigcap_{i=1}^{k} N_i = 0$.

Proof. We have $N_i = I_i M$ for some ideals I_i (i = 1, 2, …, k) of R. Then
$N_1 \; N_2 \; ... \; N_k = I_1 \; I_2 \; ... \; I_k \; M \subseteq N$, and so $I_1 I_2 ... I_k \subseteq (N:M) \subseteq \sqrt{(N:M)}$. Since $\sqrt{(N:M)}$ is a prime ideal of R, because N is a primary submodule of M, see [6], then $I_j \subseteq \sqrt{(N:M)}$ for some j = 1, 2, …, k. Therefore,
$N_j = I_j M \subseteq \sqrt{(N:M)}\, M = 0$. Thus $N_j = 0$ for some j =1, 2, …, k.
Since $\prod_{i=1}^{k} N_i \subseteq \bigcap_{i=1}^{k} N_i \subseteq N_j = 0$, then $\prod_{i=1}^{k} N_i = \bigcap_{i=1}^{k} N_i = 0$. □

Since every primary submodule is weakly primary, but the converse is not true, see Remarks 2.1, then we can generalize the previous result as follows.

Proposition 3.8 Let M be a multiplication R-module that is either
i) a P-primary R-module, or
ii) a faithful cyclic R-module.
Let N_1 , N_2 , …, N_k be submodules of M and N be a weakly primary submodule of M with $\sqrt{(N:M)}\, M = 0$. If $\prod_{i=1}^{k} N_i \subseteq N$, then $\prod_{i=1}^{k} N_i = 0$.

Proof. Note first that by Propositions 2.3 and 2.5, $\sqrt{(N:M)}$ is a weakly prime ideal of R. Now, as in the proof of Proposition 3.7, $\prod_{i=1}^{k} N_i \subseteq N$, and so $I_1 I_2 ... I_k \subseteq \sqrt{(N:M)}$. If $I_1 I_2 ... I_k \neq 0$, then the proof is similar to the proof of Proposition 3.7. So let $I_1 I_2 ... I_k = 0$, then $\prod_{i=1}^{k} N_i = I_1 I_2 ... I_k M = 0M = 0$. □

4. Direct Sum of Weakly Primary Submodules

We show in this section that the direct sum of a finite number of submodules that contain only one weakly primary submodule is weakly primary, however the direct sum of weakly primary submodules is not in general weakly primary. We start by the following result.

Proposition 4.1 Let M_1 and M_2 be unitary *R*-modules over a ring *R*. Let $M = M_1 \oplus M_2$, and $N \subseteq M_1 \oplus M_2$. Then the following are satisfied:

(1) $N = Q \oplus M_2$ is a weakly primary submodule of M if and only if Q is a weakly primary submodule of M_1.

(2) $N = M_1 \oplus Q$ is a weakly primary submodule of M if and only if Q is a weakly primary submodule of M_2.

Proof. We will prove (1) and the proof of (2) will be similar.

$\Rightarrow$) Let $N = Q \oplus M_2$ be a weakly primary submodule of *M*. Let $0 \neq rq \in Q$, $q \notin Q$. Then $(q,0) \notin Q \oplus M_2$, while $0 \neq r(q,0) \in Q \oplus M_2$. Since $N = Q \oplus M_2$ is a weakly primary submodule of M, there exists a positive integer *n* such that $r^n(M_1 \oplus M_2) \subseteq Q \oplus M_2$. Hence $r^n M_1 \subseteq Q$ for some positive integer *n*. Thus *Q* is a weakly primary submodule of M_1.

$\Leftarrow$) Suppose that Q is a weakly primary submodule of M_1. Let $0 \neq r(q,k) \in Q \oplus M_2$ with $r \in R$ and $(q,k) \in M - (Q \oplus M_2)$. Then $0 \neq rq \in Q$. Since *Q* is a weakly primary submodule of M_1, and $q \notin Q$, then $r^n M_1 \subseteq Q$ for some positive integer *n*. Thus $r^n M \subseteq Q \oplus M_2$ for some positive integer *n*. Therefore $Q \oplus M_2$ is a weakly primary submodule of *M*. □

Now, we can easily prove the following result.

Proposition 4.2 Let M_1, M_2, ... , M_n be unitary *R*-modules over a ring *R*. Let $M = M_1 \oplus M_2 \oplus ... \oplus M_n$. Q_i is a weakly primary submodule of M_i for some $i \in \{1,2,...n\}$, if and only if $(Q_i \oplus_{j=1, j\neq i}^{n} M_j)$ is a weakly primary submodule of M.

Finally, note that the direct sum of weakly primary submodules is not necessarily weakly primary, as in the following example:
Let $R = \mathbb{Z}$, the set of integers, and $M = \mathbb{Z} \oplus \mathbb{Z}$, then $N = (4) \oplus (3)$ is not a weakly primary submodule of M, since $0 \neq 3(4,1) \in N$, but neither $3^n M \subseteq N$ for any positive integer n, nor $(4,1) \in N$. However**,** (4) and (3) are weakly primary submodules of $\mathbb{Z}$.

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