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عزوم التعاميم الإحصائية الترتيبية السفلية لتوزيع قومبيرتز ومميزها

On Moments of Lower Generalized Order Statistics from Gompertz Distribution and its Characterization

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الملخص:

في هذه الدراسة سنبرهن الصيغ المحددة وبعض العلاقات المكررة للعزوم الأحادية والمضروبة للتعاميم الإحصائية الترتيبية السفلية لتوزيع قومبيرتز .

ستشمل النتائج بعض الحالات الخاصة لعلاقات العزوم للإحصاءات الترتيبية المعكوسة والبيانات السفلية. علاوة على ذلك، سنستحدم علاقة مكررة للعزوم الأحادية للحصول على مميز توزيع قومبيرتز.

Abstract:

In this paper, we establish explicit expressions and some recurrence relations for single and product moments of lower generalized order statistics from Gompertz distribution.

The results include as particular cases the above relations for moments of reversed order statistics and lower records. Further, using a recurrence relation for single moments we obtain characterization of Gompertz distribution.

Keywords: Lower generalized order statistics, order statistics, lower record values, Gompertz distribution, single and product moments and characterization

1. Introduction

The order statistics appears in many statistical applications and is widely used in statistical modeling and inference. Such models describe random variables arranged in ascending order of magnitude. In a wide subclass of generalized order statistics, representations of marginal, joint and probability density distribution functions are developed. The results are applied to obtain these representations for several expressions for the joint of generalized order statistics from Exponential Pareto distribution [9].

The Gompertz distribution plays an important role in modeling survival times, human mortality and actuarial tables. According to the literature, the Gompertz distribution was formulated by Gompertz to fit mortality tables. Examples of uses for Gompertz curves include: mobile phone uptake, where costs were initially high (so uptake was slow), followed by a period of rapid growth, followed by a slowing of uptake as saturation was reached, and population in a confined space, as birth rates first increase and then slow as resource limits are reached.

On the other hand, generalized order statistics (*GOS*) have been of interest in the past twenty years because they are more flexible in reliability theory, statistical modeling and inference, the generalized order statistics have been introduced as a unified distribution theoretical setup, which contains a variety of models of ordered random variables with different interpretations. The subject of order statistics has been further generalized and the concept of generalized order statistics (*GOS*) is introduced and studied by Kamps in a series of papers and books [1,8-10].

The ordered random variables such as order statistics play an important role in many branches of statistics and applied probability. Kamps in [6] introduced the concept of gos and showed that order statistics, record values, and some other ordered random variables can be considered as special cases of generalized order statistics. Ragab in [16] established several recurrence relations satisfied by the single and the product moments for order statistics from the Generalized Exponential Distribution (GED). The relationships can be written in terms of polygamma and hypergeometric functions and used in a simple recursive manner in order to compute the single and the product moments of all order statistics for all sample sizes, for more details see [4].

Kamps in [6] introduced the concept of generalized order statistics (gos) as a unified approach to different model e.g. usual order statistics, sequential order statistics, Stigler's order statistics and record values. They can be easily applicable in practice problems except when $F()$ is so-called inverse distribution function. For this, when $F()$ is an inverse distribution function, we need a concept of lower generalized order statistics (lgos), which is given as:

for $n \geq k, k \geq 1, m_1 = m, \dots \dots, m_n = m \in \mathrm{R}^{n-1}$, then

$$\gamma_k = k + (n-r)(m+1) > 0, 1 \leq r \leq n.$$

By the ***lgos*** from a continuous distribution function $F(x)$ with the probability density function (x) , we mean random variables $X'(1; n; m; k), \dots, X'(n; n; m; k)$ having joint pdf of the form

$$k(\prod_{r=1}^{n-1} \gamma_r)(\prod_{i=1}^{n-1}[F(x_i)]^m f(x_i))[F(x_n)]^{k-1} f(x_n) \text{ ,} \tag{1.1}$$

for $F^{-1}(1) > x_1 \geq x_2 \geq \cdots \geq x_n > F^{-1}(0)$.

The pdf of the r-th ***lgos***, is given by

$$f_{X'(r;n;m;k)}(x) = \frac{C_{r-1}}{\Gamma(r)}[F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)), \tag{1.2}$$

and the joint pdf of r-th and s-th ***lgos***, $1 \leq r < s \leq n, x > y$ 1 is

$$\begin{aligned} f_{X'(r;n;m;k),X'(s;n;m;k)}(x,y) &= \frac{C_{s-1}}{\Gamma(r)\Gamma(s-r)}[F(x)]^m f(x) g_m^{r-1}(F(x)) \\ &\times \left[h_m(F(y)) - h_m(F(x))\right]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y) \end{aligned} \tag{1.3}$$

where $C_{r-1} = \prod_{i=1}^{r} \gamma_i$ such that $\gamma_i = k + (n-i)(m+1), \Gamma(r) = (r-1)!$ and $g_m(x) = h_m(x) - h_m(1), x \in [0,1)$, such that $h_m(x) = \begin{cases} \frac{-1}{m+1} x^{m+1}, m \neq 1 \\ -lnx, \quad m = 1 \end{cases}$.

We shall take $X'(0;n;m;k)=0$ Actually if $m=0,\ k=1$ then $X'(r;n;m;k)$ reduced to the $(n-r+1)-th$ order statistics, $X_{n-r+1:n}$ from the sample $X_1,X_2,\dots,X_n$. Also, when $m=-1$, then $X'(r;n;m;k)$ reduced to the r-th lower k record value, for more details see [15].

Recurrence relations for single and product moments of lower generalized order statistics from the inverse Weibull distribution are derived by Pawlas and Szynal in [15]. Khan et al. in [8] and Khan and Kumar in [10] have established recurrence relations for moments of lower generalized order statistics from exponentiated Weibull, Pareto, gamma and generalized exponential distributions. Ahsanullah in [1], and Mbah and Ahsanullah in [13] characterized the uniform and power function distributions based on distributional properties of lower generalized order statistics, respectively.

Kamps in [7] investigated the importance of recurrence relations of order statistics in characterization. Now we will go to construct an explicit expression and some recurrence relations for single and product moments of lgos from the Gompertz distribution. Furthermore, the results for order statistics and lower record values can be deduced as special cases and a characterization of Gompertz distribution has been obtained on using a recurrence relation for single moments.

The Gompertz distribution has a continuous probability density function with location parameter a and shape parameter b,

$$f(x)=ae^{bx-\frac{a}{b}\left(e^{bx-1}\right)},x\geq 0\ , \tag{1.4}$$

and the corresponding distribution function is

$$F(x)=1-e^{-\frac{a}{b}\left(e^{bx-1}\right)}\ . \tag{1.5}$$

In addition, we can get a relationship between $f(x)$ and $F(x)$ from (1.4) and (1.5) that can be written as

$$F(x)=1-\frac{f(x)}{ae^{bx}}\ . \tag{1.6}$$

2. Explicit expression for single moments

In this section we reduced $X'(r,n,m,k)$ to the r-th lower k record value when $m=-1$ in the explicit expression for single moments and also get it in general when $m \neq 1$, for more details see [15].

Case 1: $m \neq 1$

The single moments of **lgos** for Gompertz distribution can be obtained from (1.1), (1.2) and (1.5) as follows:

$$E\left[X'^{j}_{(r,n,m,k)}\right]=\frac{C_{r-1}}{\Gamma(r)}\int_0^\infty x^j[F(x)]^{\gamma_s-1}f(x)g_m^{r-1}\big(F(x)\big)dx$$

$$=\frac{C_{s-1}}{(m+1)^{r-1}\Gamma(r)}\int_0^\infty x^j\left[1-e^{-\frac{a}{b}(e^{bx-1})}\right]^{\gamma_s-1}ae^{bx-\frac{a}{b}(e^{bx-1})}\times\left\{1-\left[e^{-\frac{a}{b}(e^{bx-1})}\right]^{m+1}\right\}^{r-1}dx \tag{2.1}$$

Expanding $\left\{1-\left[1-e^{-\frac{a}{b}(e^{bx-1})}\right]^{m+1}\right\}^{r-1}$ binomially, we have

$$E\left[X'^{j}_{(r,n,m,k)}\right]=\frac{C_{r-1}}{(m+1)^{r-1}\Gamma(r)}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\times\int_0^\infty x^j\left[1-e^{-\frac{a}{b}(e^{bx-1})}\right]^{\gamma_{r-i}-1}ae^{bx-\frac{a}{b}(e^{bx-1})}dx.$$

Using the transformation, $t=-lnF(x)$, we obtain

$$E\left[X'^{j}_{(r,n,m,k)}\right]=\frac{b^{-j}C_{r-1}}{(m+1)^{r-1}\Gamma(r)}\sum_{i=0}^{r-1}\sum_{|v|=0}^{i}(-1)^{i+v}\frac{b^v}{a^v}\binom{r-1}{i}\binom{j}{v}\frac{\Gamma(1+v)}{(\gamma_{r-1}+1)^{1+v}}, \tag{2.2}$$

for more details see [3].

Case 2: $m=1$

$$E\left[X'^{j}_{(r,n-1,k)}\right]=\frac{k^{-j}b^{-j}}{\Gamma(r)}\Gamma(r+j) \tag{2.3}$$

Special cases

i) Putting $m=0, k=1 \ in \ (2.2)$, the explicit formula for single moments of order statistics of the Gompertz distribution can be obtained as

$$E\left[X'^{j}_{(n-r+1:n)}\right] = b^{-j} C_{r:n} \sum_{i=0}^{r-1} \sum_{|v|=0}^{j} (-1)^{i+v} \frac{b^v}{a^v} \binom{r-1}{i} \binom{j}{v} \frac{\Gamma(1+v)}{(n-r+i+2)^{1+v}}, \quad (2.4)$$

where,

$$C_{r:n} = \frac{n!}{(r-1)!\,(n-r)!}.$$

ii) Putting $k=1 \ in \ (2.3)$, we deduce the explicit expression for the moments of lower record values for the Gompertz distribution as

$$E\left[X^{j}_{L(r)}\right] = \frac{b^{-j}}{\Gamma(r)} \Gamma(r+j). \quad (2.5)$$

3. Explicit expression for product moments

In this section we reduced $X'(r,n,m,k)$ to the $r-th$ lower k record value when $m=-1$ in the explicit expression for product moments and also get it in general when $m \neq -1$, for more details see [15].

Case 1: $m \neq -1$

On using (1.3) and binomial expansion, the the explicit expressions for the product moments of ***lgos*** $X'^{i}(r,n,m,k)$ and $X'^{j}(s,n,m,k)$, $1 \le r < s \le n$ can be obtained when $m \neq -1$, as

$$E\left[X'^{i}_{(r,n,m,k)} X'^{j}_{(s,n,m,k)}\right] = \frac{C_{s-1}}{\Gamma(r)\Gamma(s-r)(m+1)^{s-2}} \sum_{\alpha=0}^{r-1} \sum_{\beta=0}^{s-r-1} \binom{r-1}{\alpha} \binom{s-r-1}{\beta} \times \int_0^\infty y^j [F(y)]^{\gamma_{s-\beta}-1} f(y) I(y) dy \quad (3.1)$$

where,

$$I(y) = \int_0^\infty x^j [F(x)]^{(s-r+\alpha-\beta)(m+1)-1} f(x) dx \quad (3.2)$$

by setting $t = -lnF(x) \ in$ (3.2), we get

$$I(y) = \sum_{c=0}^{i} \binom{i}{c} \frac{1}{b^i[(s-r+\alpha-\beta)(m+1)]^{1+c}} IG[1 + c, (s-r+\alpha-\beta)(m+1)(-lnF(x))]$$

$$= \sum_{c=0}^{i} \binom{i}{c} \frac{1}{b^i[(s-r+\alpha-\beta)(m+1)]^{1+c}}$$

$$\times \sum_{z=0}^{\infty} (-1)^z \frac{(s-r+\alpha-\beta)(m+1)(-lnF(x))^z}{z!\,(z+1+c)}$$

$$= b^{-i} \sum_{c=0}^{\infty} \binom{i}{c} \sum_{z=0}^{\infty} (-1)^z \frac{[(s-r+\alpha-\beta)(m+1)]^z (-lnF(x))^{z+1+c}}{z!(z+1+c)} \tag{3.3}$$

where $IG(..,..)$ denotes the incomplete gamma function defined by

$$IG(l,w) = \int_0^w u^{l-1} e^{-u} du.$$

On substituting, the above expression of $I(y)$ in (3.1), we will get

$$\boldsymbol{E}\left[\boldsymbol{X}'^{\boldsymbol{i}}_{(\boldsymbol{r},\boldsymbol{n},\boldsymbol{m},\boldsymbol{k})} \boldsymbol{X}'^{\boldsymbol{j}}_{(\boldsymbol{s},\boldsymbol{n},\boldsymbol{m},\boldsymbol{k})}\right] = \frac{b^{-i} C_{s-1}}{\Gamma(r)\Gamma(s-r)(m+1)^{s-2}} \sum_{\alpha=0}^{r-1} \sum_{\beta=0}^{s-r-1} \sum_{c=0}^{i} \sum_{z=0}^{\infty} (-1)^{\alpha+\beta+z} \times \binom{r-1}{\alpha} \binom{s-r-1}{\beta} \binom{i}{c} \frac{[(s-r+\alpha-\beta)(m+1)]^z}{z!(z+1+c)} \times \int_0^{\infty} y^j \, [F(y)]^{\gamma_s-\beta-1} f(y) [lnF(x)]^{z+1+c} dy. \tag{3.4}$$

Again setting $t = -lnF(x)\ in$ (3.1) and then simplifying the resulting equation to obtain that

$$\boldsymbol{E}\left[\boldsymbol{X}'^{\boldsymbol{i}}_{(\boldsymbol{r},\boldsymbol{n},\boldsymbol{m},\boldsymbol{k})} \boldsymbol{X}'^{\boldsymbol{j}}_{(\boldsymbol{s},\boldsymbol{n},\boldsymbol{m},\boldsymbol{k})}\right] = \frac{b^{-(i+j)} C_{s-1}}{\Gamma(r)\Gamma(s-r)(m+1)^{s-2}} \sum_{\alpha=0}^{r-1} \sum_{\beta=0}^{s-r-1} \sum_{c=0}^{i} \sum_{z=0}^{\infty} (-1)^{\alpha+\beta+z} \times \binom{r-1}{\alpha} \binom{s-r-1}{\beta} \binom{i}{c} \binom{j}{d} \frac{[(s-r+\alpha-\beta)(m+1)]^z}{z!(z+1+c)} \times \frac{\Gamma(z+2+c+d)}{(\gamma_s-\beta+1)^{z+2+c+d}} \tag{3.5}$$

Case 2: $m = -1$

$$E\left[X'^{i}_{(r,n,-1,k)} X'^{j}_{(s,n,-1,k)}\right] = \frac{(kb)^{-(i+j)} C_{s-1}}{\Gamma(r)\Gamma(s-r)} \sum_{\alpha=0}^{s-r-1} (-1)^{s-r-\alpha-1} \binom{s-r-1}{\alpha} \frac{\Gamma(s+i+j)}{(s-\alpha-1+i+j)} \tag{3.6}$$

Special cases

i) Putting $m = 0, k = 1 \, in$ (3.5), the explicit formula for product moments of order statistics of the Gompertz distribution can be obtained as

$$\boldsymbol{E}\left[\boldsymbol{X}'^{i}_{(r,n,m,k)}\boldsymbol{X}'^{j}_{(s,n,m,k)}\right] = \frac{b^{-(i+j)}C_{r,s:n}}{\Gamma(r)\Gamma(s-r)(m+1)^{s-2}}\sum_{\alpha=0}^{r-1}\sum_{\beta=0}^{s-r-1}\sum_{c=0}^{i}\sum_{z=0}^{\infty}(-1)^{\alpha+\beta+z}\times \binom{r-1}{\alpha}\binom{s-r-1}{\beta}\binom{i}{c}\binom{j}{d}\frac{[(s-r+\alpha-\beta)(m+1)]^{z}\Gamma(z+2+c+d)}{z!(z+1+c)(n-s+1+\beta)^{z+2+c+d}} \qquad (3.7)$$

where,

$$C_{r:n} = \frac{n!}{(r-1)!\,(n-r+1)!\,(n-s)!}.$$

ii) Putting $k = 1 \, in$ (3.6), we deduce the explicit expression for the product moments of lower record values for the Gompertz distribution as

$$E\left[X^{i}_{L(r)}, X^{j}_{L(s)}\right] = \frac{(b)^{-(i+j)}}{\Gamma(r)\Gamma(s-r)}\sum_{\alpha=0}^{s-r-1}(-1)^{s-r-\alpha-1}\binom{s-r-1}{\alpha}\frac{\Gamma(s+i+j)}{(s-\alpha-1+i+j)}. \qquad (3.8)$$

4. Characterization of Gompertz distribution

The problem of characterizing a distribution is an important problem, which has recently attracted the attention of many researchers. In this section, a characterization of Gompertz distribution has been obtained on using a recurrence relation for single moments.

$\boldsymbol{Theorem}\ \mathbf{4.1}$. Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0) = 0 \ \ and \ \ 0 < F(x) < 1$ for all $x > 0$, then

$$\boldsymbol{E}\left[\boldsymbol{X}'^{j-1}_{(r,n,m,k)}\right] = \frac{1}{j}\Big([b+\gamma_r]E\left[\boldsymbol{X}'^{j}_{(r,n,m,k)}\right] + (r-1)E\left[\boldsymbol{X}'^{j}_{(r-1,n,m,k)}\right] + E\left[\boldsymbol{a}\boldsymbol{e}^{\boldsymbol{bx}}\boldsymbol{X}'^{j}_{(r,n,m,k)}\right]\Big) \qquad (4.1)$$

if and only if

$$F(x) = 1 - e^{-\frac{a}{b}(e^{bx-1})},$$

$Proof.$ If the recurrence relation in equation (4.1) is satisfied, then on using equation (2.1), we have

$$\frac{C_{r-1}}{\Gamma(r)}\int_0^\infty x^{j-1}\,[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}[F(x)]dx$$
$$=\frac{bC_{r-1}}{j\Gamma(r)}\int_0^\infty x^j\left(F(x)\right)^{\gamma_r-1}f(x)g_m^{r-1}[F(x)]dx$$
$$+\frac{\gamma_r C_{r-1}}{j\Gamma(r)}\int_0^\infty x^j\left(F(x)\right)^{\gamma_r-1}f(x)g_m^{r-1}[F(x)]dx$$
$$+\frac{(r-1)C_{r-1}}{j\Gamma(r)}\int_0^\infty x^j\left(F(x)\right)^{\gamma_r+m}f(x)g_m^{r-2}[F(x)]dx$$
$$+\frac{C_{r-1}}{j\Gamma(r)}\int_0^\infty x^j\,ae^{bx}\left(F(x)\right)^{\gamma_r+m}f(x)g_m^{r-2}[F(x)]dx$$
$$=\frac{C_{r-1}}{j\Gamma(r)}\int_0^\infty x^j\left(F(x)\right)^{\gamma_r-1}f(x)g_m^{r-1}[F(x)]$$
$$\times\left[b+\gamma_r+\frac{(r-1)\left(F(x)\right)^m}{g_m\left(F(x)\right)}+ae^{bx}\right]dx \tag{4.2}$$

Integrating the right hand side in (4.2) by parts, we obtain

$$\frac{C_{r-1}}{\Gamma(r)}\int_0^\infty x^{j-1}\,ae^{bx}\left(1-F(x)\right)[F(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]dx$$
$$=\frac{C_{r-1}}{\Gamma(r)}\int_0^\infty x^{j-1}\,ae^{bx}\left(1-F(x)\right)[F(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]dx \tag{4.3}$$

which reduces to

$$\frac{C_{r-1}}{\Gamma(r)}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}[F(x)]\{ae^{bx}[F(x)-1]+f(x)\}dx \tag{4.4}$$

Now applying a generalization of the Muntz-Szasz, $Theorem$ [4.1] to equation (4.4), we get

$ae^{bx}([F(x)]-1)=-f(x)$

which prove that

$$F(x)=1-e^{-\frac{a}{b}\left(e^{bx-1}\right)}.$$

5. Conclusion

This paper deals with the lower generalized order statistics from the Gompertz distribution. Recurrence relations between the single and product moments are derived. Characterization of the Gompertz distribution based on a recurrence relation for single moments is discussed.

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