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Some Results On Almost 2-Absorbing Submodules

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Abstract:.

Let R be a commutative ring with identity and M be a unitary R-module, In this paper, we investigate some properties of almost 2-absorbing submodules of M as a new generalization of 2-absorbing and weakly 2-absorbing submodules. We study some basic properties of almost 2-absorbing submodules and give some characterizations of them, especially for (finitely generated faithful) multiplication modules.

Key words and phrases: 2-absorbing submodules; almost 2-absorbing submodules; multiplication modules.

1. Introduction:

Throughout this article, we consider all rings as commutative rings with identity and all modules as unital. For any two submodules N and K of an R-module M, the residual of N by K is defined as the set $(N:K)=\{r\in R: rM\subseteq N\}$ which is clearly an ideal of R. In particular, the ideal $(0:M)$ is called the annihilator of M and is denoted by *Ann zero* ideal of R. Let N be a submodule of M and I be an ideal of R. The residual submodule of N by I is defined as $(N:I)=\{m\in M: I(m)\subseteq N\}$. These two residual ideal and submodule were proved to be useful in studying many concepts of modules, see, for example, [2, 4, 7].

Recall that a proper submodule N of an R-module M is a 2-absorbing (resp. weakly 2-absorbing) submodule of M [3] if, whenever $abm\in N$ (resp. $0\neq abm\in N$) for $a,b\in R$ and $m\in M$ then $am\in N$ or $bm\in N$ or $ab\in(N:M)$. A proper submodule K is maximal in M if there is no proper submodule of M containing K properly. A local module is a module with unique maximal submodule. An R-module M is called a multiplication module provided that, for every submodule N of M, there exists an ideal I of R so that $N=IM$ (or equivalently, $N=(N:M)M$). Multiplication modules were studied extensively in [4, 6, 8]. An R-module M is called a cancellation module of R if, for all ideals I and J of R, $IM=JM$ implies that $I=J$.

The class of prime submodules of modules was introduced and studied in 1992 as a generalization of the class of prime ideals of rings. Then, many generalizations of prime submodules were studied such as primary, primal, classical prime, weakly prime, 2-absorbing and weakly 2-absorbing . In this article, we study almost 2-absorbing submodules as one of the generalizations of 2-absorbing (and weakly 2-absorbing) submodules. We generalize some basic properties of 2-absorbing and weakly 2-absorbing to almost 2-absorbing submodules.

In particular, we give characterizations of almost 2-absorbing submodules in multiplication modules.

2. Some Properties of Almost 2-absorbing Submodules

Definition: [9] A proper ideal I of R is called an almost 2-absorbing ideal if $a,b,c \in R$ with $abc \in I - I^2$ implies that $ab \in I$ or $ac \in I$, or $bc \in I$.

Definition: [9] *A proper submodule N of R-module M is called an almost 2-absorbing submodule of M if, whenever* $a,\ b \in R$ *and* $m \in M$ *such that* $abm \in N - (N : M)N$ *, implies that* $ab \in (N : M)$ *or* $am \in N$ *or* $bm \in N$.

Clearly, any 2-absorbing submodule is weakly 2-absorbing and any weakly 2-absorbing submodule is almost 2-absorbing. However, the converses are not necessarily true. For example, let $R = Z$, $M = Z_{48}$ and let $N = < 16 >$, then $(N : M)N = 16Z < 16 > = < 16 >$ and hence N is almost 2-absorbing submodule, but N is not a 2-absorbing since $2.2.4 \in < 16 >$ but $2.4 \notin < 16 >$ and $2.2 \notin (N : M)$.

Theorem 2.1: *Let* N, K *be R-submodules of M with* $K \subseteq N$ *. If N is an almost 2-absorbing submodule of M then* N / K *is an almost 2-absorbing R-submodule of* M / K .

Proof: Let $a,\ b \in R$ and $m + K \in M / K$ such that $ab(m + K) \in (N / K) - (N / K : M / K)N / K$. Since

$(N : M) = (N/K : M/K)$ then, $abm + K \in N/K - (N : M)N/K$ and so $abm \in N - (N : M)N$. As N is almost 2-absorbing in M, then $am \in N$ or $bm \in N$ or $ab \in (N : M)$. Therefore, $a(m + K) \in N/K$ or $b(m + K) \in N/K$ or $ab \in (N/K : M/K)$, and hence N/K is almost 2-absorbing in M/K.

Proposition 2.2: *Let N be an almost 2-absorbing submodule of R-module M. If S is a multiplicatively closed subset of R, then $S^{-1}N$ is almost 2-absorbing submodule in R-module $S^{-1}M$.*

Proof: Let $a, b \in R$, $s \in S$ and $m \in M$ such that $ab(\frac{m}{s}) \in S^{-1}N - (S^{-1}N : S^{-1}M)S^{-1}N$. Then, $\frac{abm}{s} \in S^{-1}N - S^{-1}((N : M)N)$. Indeed, if $\frac{abm}{s} \in S^{-1}((N : M)N)$, then there is $t \in S$ such that $\frac{abm}{s} = \frac{r_1n_1 + r_2n_2 + \ldots + r_kn_k}{t} = r_1\frac{n_1}{t} + r_2\frac{n_2}{t} + \ldots + r_k\frac{n_k}{t}$ where $r_i \in (N : M)$ and $n_i \in N$, $i = 1, 2, \ldots, k$. Thus, $\frac{abm}{s} = \in (N : M)(S^{-1}N) \subseteq (S^{-1}N : S^{-1}M)S^{-1}N$, which is a contradiction. As $\frac{abm}{s} \in S^{-1}N$, there is $t \in S$, such that $ab(tm) \in N - (N : M)N$, since N is almost 2-absorbing then $a(tm) \in N$ or $b(tm) \in N$ or $ab \in (N : M) \subseteq (S^{-1}N : S^{-1}M)$

and hence $\frac{atm}{ts} = a\frac{m}{s} \in S^{-1}N$ or $\frac{btm}{ts} = b\frac{m}{s} \in S^{-1}N$ or $ab \in \left(S^{-1}N : S^{-1}M\right)$.

Proposition 2.3: *Let Q be a submodule of R-module M, N be an any R-module. Then Q is an almost 2-absorbing submodule of M if and only if $Q \oplus N$ is an almost 2-absorbing submodule of $M \oplus N$.*

Proof: (→) Suppose Q is almost 2-absorbing submodule in M. Let $a, b \in R$, $(m, n) \in M \oplus N$ such that $ab(m, n) \in (Q \oplus N) - (Q \oplus N : M \oplus N)(Q \oplus N)$, then we get $abm \in Q - (Q : M)Q$, by $(Q \oplus N : M \oplus N) = (Q : M)$. Since Q is almost 2-absorbing submodule in M, then $am \in Q$ or $bm \in Q$ or $ab \in (Q : M)$, that is, $a(m, n) \in Q \oplus N$ or $b(m, n) \in Q \oplus N$ or $ab \in (Q \oplus N : M \oplus N)$. So $Q \oplus N$ is almost 2-absorbing submodule in $M \oplus N$.

(←) Suppose $Q \oplus N$ is almost 2-absorbing submodule in $M \oplus N$. Let $a, b \in R$, $m \in M$ such that $abm \in Q - (Q : M)Q$. Then we get $ab(m, 0) \in (Q \oplus N) - (Q \oplus N : M \oplus N)(Q \oplus N)$. Since $Q \oplus N$ is almost 2-absorbing, then $ab \in (Q \oplus N : M \oplus N)$ or $a(m, 0) \in Q \oplus N$ or $b(m, 0) \in Q \oplus N$, that is, $am \in Q$ or $bm \in Q$ or $ab \in (Q : M)$. Hence, Q is almost 2-absorbing submodule in M.

If N is a submodule of R-module M and r ∈ R then a submodule N_r of M is defined by $N_r = (N : r) = \{m \in M : rm \in N\}$.

Theorem 2.4: Let M be an R-module and N be a proper submodule of M. The following are equivalent :

(1) N is an almost 2-absorbing submodule.

(2) For $a, b \in R$ such that $ab \notin (N : M)$, $N_{ab} = N_a \cup N_b \cup [(N : M)N]_{ab}$.

Proof: (1 → 2) Let N be an almost 2-absorbing submodule, and assume that $ab \notin (N : M)$, let $m \in N_{ab}$ then $abm \in N$. If $abm \notin (N : M)N$ then $am \in N$ or $bm \in N$ and hence $m \in N_a$ or $m \in N_b$. If $abm \in (N : M)N$ then $m \in [(N : M)N]_{ab}$. The other containment holds for any submodule.

(2 → 1) Let $a, b \in R$ and m ∈ M with $abm \in N - (N : M)N$. Assume that $ab \notin (N : M)$ then $m \in N_{ab} = N_a \cup N_b \cup [(N : M)N]_{ab}$, but $abm \notin (N : M)N$ then $m \in N_a$ or $m \in N_b$, thus $am \in N$ or $bm \in N$.

Proposition 2.5: Let M be an R-module and N be a proper submodule of M, then N is an almost 2-absorbing submodule in M if and only if for any $a, b \in R$ and submodule K of M such that $abK - \{0\} \subseteq N - (N : M)N$, we have $ab \in (N : M)$ or $aK \subseteq N$ or $bK \subseteq N$.

Proof: (→) Assume that $ab \notin (N : M)$, then $K \subseteq N_{ab} = N_a \cup N_b \cup [(N : M)N]_{ab}$, but $abK \not\subset (N : M)N$ then $K \subseteq N_a$ or $K \subseteq N_b$ and hence $aK \subseteq N$ or $bK \subseteq N$.

(←) Suppose that $abm \in N - (N : M)N$ for $a, b \in R$ and m ∈ M. Then, $ab(m) - \{0\} \subseteq N - (N : M)N$ and so $ab \in (N : M)$ or $a(m) \subseteq N$ or $b(m) \subseteq N$. Therefore, $ab \in (N : M)$ or $am \in N$ or $bm \in N$, thus N is almost 2-absorbing.

Theorem 2.6: Let M be an R-module and N be a proper submodule of M, then N is almost 2-absorbing submodule in M if and only if $N / (N : M)N$ is weakly 2-absorbing submodule in $M / (N : M)N$.

Proof: (→) Suppose N is almost 2-absorbing submodule in M. Let $a, b \in R$, m ∈ M such that $0 \neq ab(m + (N : M)N) \in N / (N : M)N$. Then $abm \in N - (N : M)N$, and so $am \in N$ or $bm \in N$ or $ab \in (N : M)$ since N is almost 2-absorbing. So $ab \in (N/(N : M)N : M/(N : M)N) =$ $(N : M)$ or $a(m + (N : M)N) \in N/(N : M)N$ or $b(m + (N : M)N)$ $\in N/(N : M)N$. Hence $N/(N : M)N$ is weakly 2-absorbing in $M/(N : M)N$.

(←) Assume $N/(N : M)N$ is weakly 2-absorbing in $M/(N : M)N$. Let $a, b \in R$, $m \in M$ such that $abm \in N - (N : M)N$. So $0 \neq ab(m + (N : M)N) \in N/(N : M)N$. Then we have ab ∈ $(N/(N : M)N : M/(N : M)N) = (N : M)$ or $a(m + (N : M)N)$ $\in N/(N : M)N$ or $b(m + (N : M)N) \in N/(N : M)N$. That is , $am \in N$ or

$bm \in N$ or $ab \in (N : M)$. Hence N is almost 2-absorbing submodule in M.

3. Almost 2-absorbing submodule of multiplication modules

If M is a multiplication R-module and $N = IM$, $K = JM$ are two submodules of M, then the product NK of N and K is defined as $NK = (IM)(JM) = (IJ)M$. In particular, we have $N^2 = ((N : M)M)((N : M)M) = (N : M)^2 M$. Let N be a submodule of R-module M, N is called idempotent in M if $(N : M)N = N$. Obviously, every idempotent submodule is almost 2-absorbing submodule in M. Before giving Theorem 3.2 we need the following Lemma.

Lemma 3.1: [5] Let N be a submodule of a finitely generated faithful multiplication (and so cancellation) R-module M. Then, we have $(IN : M) = I(N : M)$ for every ideal I of R.

Theorem 3.2: Let M be a finitely generated faithful multiplication R-module and N be a proper submodule of M. The following are equivalent :

(1) N is almost 2-absorbing submodule in M.

(2) (N : M) is almost 2-absorbing ideal in R.

(3) N = QM for some almost 2-absorbing ideal Q of R.

Proof: (1 → 2) Suppose N is almost 2-absorbing submodule and let $a, b, c \in R$ such that $abc \in (N : M) - (N : M)^2$. Then, $abcM - \{0\} \subseteq N - (N : M)N$. Indeed, if $abcM \subseteq (N : M)N$, then by Lemma (3.1), $abc \in ((N : M)N : M) = (N : M)^2$, a contradiction. Now , since N is almost 2-absorbing submodule implies that $ab \in (N : M)$ or $acM \subseteq N$ or $bcM \subseteq N$ (and so

$ac \in (N : M)$ or $ab \in (N : M)$). Hence, $(N : M)$ is almost 2-absorbing ideal in R.

$(2 \to 1)$ Let $a, b \in R$ and $m \in M$, such that $abm \in N - (N : M)N$. Then, $ab((m) : M) \subseteq ((abm) : M) \subseteq (N : M)$.Moreover, $ab((m) : M) \not\subset (N : M)^2$ because otherwise, if $ab((m) : M) \subseteq (N : M)^2 \subseteq ((N : M)N : M)$, then, $ab(m) = ab((m) : M)M \subseteq (N : M)N$, a contradiction. As $(N : M)$ is almost 2-absorbing ideal in R, then, $ab \in (N : M)$ or $a((m) : M) \subseteq (N : M)$ or $b((m) : M) \subseteq (N : M)$. In the second case, we obtain $(am) \subseteq a(m) = a((m) : M)M \subseteq (N : M)M = N$ and so $am \in N$, similarly we have $bm \in N$. Thus N is an almost 2-absorbing submodule in M.

$(2 \leftrightarrow 3)$ We choose Q = (N : M).

Proposition 3.3: Let M be a local multiplication R-module with a unique maximal submodule Q and $(Q : M)Q = 0$, then any proper submodule of M is almost 2-absorbing if and only if it is weakly 2-absorbing .

Proof: $(\to)$ For any proper submodule N of M, $N \subsetneq Q$, $(N : M)N = 0$, because $(Q : M)Q = 0$. Whenever $a, b \in R$, m $\in$ M such that $abm \in N - (N : M)N$ we have $0 \neq abm \in N$. So if N is almost 2-absorbing, then it is weakly 2-absorbing in M.

$(\leftarrow)$ It is trivial, because any weakly 2-absorbing submodule is almost 2-absorbing.

Lemma 3.4: [1] Let N be a submodule of a faithful multiplication R-module M and I be a finitely generated faithful multiplication ideal of R. Then,

(1) $N = (IN : I)$.

(2) If $N \subseteq IM$, then $(JN : I) = J(N : I)$ for any ideal J of R.

(3) $(N : I) = ((N : M) : I)M = (N : IM)M$.

Theorem 3.5: Let N be a submodule of a faithful multiplication R-module M and I be a finitely generated faithful multiplication ideal of R. Then, N is an almost 2-absorbing submodule of IM if and only if $(N : I)$ is almost 2-absorbing in M.

Proof: Suppose that N is almost 2-absorbing submodule in IM. Let $a, b \in R$ and $m \in M$, such that $abm \in (N : I) - ((N : I) : M)(N : I)$. Then, $abIm - \{0\} \subseteq N - (N : IM)N$. In fact, if $abIm \subseteq (N : IM)N$, then by Lemma 3.4, $abm \in ((N : IM)N : I) = (N : IM)(N : I) = ((N : I) : M)(N : I)$, a contradiction. As N is almost 2-absorbing submodule in IM, then, $aIm \subseteq N$ or $bIm \subseteq N$ or $ab \in (N : IM)$. If $aIm \subseteq N$ or $bIm \subseteq N$, then, $am \in (N : I)$ or $bm \in (N : I)$. Suppose $ab \in (N : IM)$, so that $abIM \subseteq N$. Then again by Lemma 3.4, $abM = ab(IM : I) \subseteq (abIM : I) \subseteq (N : I)$, and so, $ab \in ((N : I) : M)$. Therefore, (N : I) is almost 2-absorbing submodule in M.

Conversely, suppose that (N : I) is almost 2-absorbing submodule in M. Let $a, b \in R$ and K be a submodule of IM such that $abK - \{0\} \subseteq N - (N : IM)N$. Then ,

$ab(K:I)\subseteq(abK:I)\subseteq(N:I)$. Moreover, if $ab(K:I)\subseteq((N:I):M)(N:I)=(N:IM)(N:I)$, then,
$abK=ab(IK:I)=ab(K:I)I\subseteq(N:IM)(N:I)I=(N:IM)N$, a contradiction. As $(N:I)$ is almost 2-absorbing submodule in M, then $ab\in((N:I):M)=(N:IM)$ or $a(K:I)\subseteq(N:I)$ or $b(K:I)\subseteq(N:I)$, which implies that $aK=a(K:I)I\subseteq a(N:I)N\subseteq N$ or
$bK=b(K:I)I\subseteq b(N:I)N\subseteq N$. Hence, N is almost 2-absorbing submodule in IM .

Theorem 3.6: Let M be a finitely generated faithful multiplication R-module and P be a proper submodule of M, then P is almost 2-absorbing submodule in M if and only if whenever N, K and L are submodules of M such that $NKL-\{0\}\subseteq P-(P:M)P$, we have $NK\subseteq P$ or $NL\subseteq P$ or $KL\subseteq P$.

Proof: ($\rightarrow$) Suppose P is almost 2-absorbing submodule in M. So by theorem 3.2, (P : M) is almost 2-absorbing ideal in R. We have $N=(N:M)M$, $K=(K:M)M$ and $L=(L:M)M$. Then $NKL=(N:M)(K:M)(L:M)M$. Suppose $NKL-\{0\}\subseteq P-(P:M)P$, but $NK\not\subset P$, $NL\not\subset P$ and $KL\not\subset P$. Then $(N:M)(K:M)\not\subset(P:M)$, $(N:M)(L:M)\not\subset(P:M)$ and $(K:M)(L:M)\not\subset(P:M)$. As $(P:M)$ is almost 2-absorbing in R, so $(N:M)(K:M)(L:M)\not\subset(P:M)$ or $(N:M)(K:M)(L:M)\subseteq(P:M)^2$. If

$(N : M)(K : M)(L : M) \not\subset (P : M)$, then $NKL = (N : M)(K : M)(L : M)M \not\subset (P : M)M = P$, which is a contradiction. If $(N : M)(K : M)(L : M) \subseteq (P : M)^2$, then $NKL = (N : M)(K : M)(L : M)M \subseteq (P : M)^2 M = (P : M)P$, which is a contradiction. Therefore, $NK \subseteq P$ or $NL \subseteq P$ or $KL \subseteq P$.

(←) To prove that P is almost 2-absorbing submodule in M, by theorem 3.2 it is enough to prove that (P : M) is almost 2-absorbing ideal in R. Let $a, b, c \in R$ such that $abc \in (P : M) - (P : M)^2$, then $abcM - \{0\} \subseteq P - (P : M)P$. Take $aM = N$, $bM = K$ and $cM = L$, we get $NKL - \{0\} \subseteq P - (P : M)P$. By assumption, $NK \subseteq P$ or $NL \subseteq P$ or $KL \subseteq P$; that is, $aM \subseteq P$ or $acM \subseteq P$ or $bcM \subseteq P$. Hence $ab \in (P : M)$ or $ac \in (P : M)$ or $bc \in (P : M)$. Thus (P : M) is almost 2-absorbing ideal in R; that is, P is almost 2-absorbing submodule in M.

Corollary 3.7: Let P be a proper submodule of a finitely generated faithful multiplication R-module M, then P is almost 2-absorbing submodule in M if and only if whenever $x, y, z \in M$ such that $xyz \in P - (P : M)P$, we have $xy \in P$ or $xz \in P$ or $yz \in P$.

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