# On Cerone's Generalizations of Steffensen Type Inequality

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**Abstract:**

*In this article, we state and prove new extensions of some Steffensen type inequalities which allow bounds involving any two subintervals instead of restricting them to include the end points motivated by Cerone . We show that Cerone's results, which have been previously published, can be obtained as special cases of those obtained here.*

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## 1. Introduction

Inequalities are at the heart of mathematical analysis [2, 4, 6]. Steffensen's integral inequality was established in 1918 and it lies in the core of integral inequalities. It can be used for dealing with the comparison between integrals over a whole interval $[a, b]$ and integrals over a subinterval [7].

The following inequality is well- known in the literature as Steffensen's inequality[11]

**Theorem 1.1** Assume that two integrable functions $f(t)$ and $g(t)$ are defined on the interval $[a,b]$ with $f(t)$ nonincreasing and that $0 \le g(t) \le 1$ on $[a,b]$. Then

$$\int_{b-\lambda}^{b} f(t)\,dt \le \int_{a}^{b} f(t)\,g(t)\,dt \le \int_{a}^{a+\lambda} f(t)\,dt \qquad (1.1)$$

where $\lambda = \int_{a}^{b} g(t)\,dt$. (1.2)

If $f$ is non-decreasing, then (1.1) is reversed .

Steffensen's inequality has been generalized in many ways. In this section, we focus at Cerone's generalizations of Steffensen's inequality. The following theorem contains some of Cerone's generalizations (see [3]).

**Theorem 1.2 [3]** Let $f, g : [a,b] \to \mathbb{R}$ be integrable functions on $[a,b]$ and let $f$ be nonincreasing. Further, let $0 \le g(t) \le 1$ and $\lambda = \int_{a}^{b} g(t)dt = d_i - c_i$, where $[c_i, d_i] \subset [a,b]$ for $i = 1, 2$ and $d_1 \le d_2$.

**(a)** Then

$$\int_{a}^{b} f(t)g(t)\,dt \le \int_{c_1}^{d_1} f(t)\,dt + R(c_1, d_1) \qquad (1.3)$$

holds where,

$$R(c_1, d_1) = \int_{a}^{c_1} \left( f(t) - f(d_1) \right) g(t)\,dt \ge 0. \qquad (1.4)$$

**(b)** Then

$$\int_{c_2}^{d_2} f(t)\,dt - r(c_2, d_2) \le \int_{a}^{b} f(t)g(t)\,dt \qquad (1.5)$$

holds where,

$$r(c_2, d_2) = \int_{d_2}^{b} \left( f(c_2) - f(t) \right) g(t)\,dt \ge 0. \qquad (1.6)$$

**Remark 1.1** If in Theorem 1.2 we take $c_1 = a$ (and so $d_1 = a + \lambda$), then $R(a, a+\lambda) = 0$. Further, taking $d_2 = b$ so that $c_2 = b - \lambda$, gives $r(b-\lambda, b) = 0$ . The Steffensen's inequality (1.1) is thus recaptured. Since (1.2) holds, then $c_2 \geq a$ and $d_1 \leq b$ giving $[c_i, d_i] \subset [a, b]$. Theorem 1.2 may thus be viewed as a generalization of the Steffensen's inequality as given in Theorem 1.1, to allow for two equal length subintervals that are not necessarily at the ends of $[a, b]$.

In our paper [ 1 ], we extended the results of Theorem 1.2 for the case $i = 1, 2, \ldots, n$ and established the following theorem .

**Theorem 1.3 [ 1 ]** Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions on $[a, b]$ and let $f$ be negative and nonincreasing. Further, let $0 \leq g(t) \leq 1$ and $\lambda = \int_a^b g(t)dt = d_i - c_i$ ,where $[c_i, d_i] \subset [a,b]$ for $i = 1, 2, \ldots, n$ and $d_1 \leq d_2 \leq \ldots \leq d_n$, $n = 2, 3, 4, \ldots$ .

**(a)** Then

$$(n-1)\int_a^b f(t)g(t)\,dt \leq \int_{c_1}^{d_1} f(t)\,dt + \int_{c_2}^{d_2} f(t)\,dt + \ldots + \int_{c_{n-1}}^{d_{n-1}} f(t)\,dt + R(c_{n-1}, d_{n-1}) \tag{1.7}$$

holds where,

$$R(c_{n-1}, d_{n-1}) = \int_a^{c_1} \big(f(t) - f(d_1)\big) g(t)\,dt + \int_a^{c_2} \big(f(t) - f(d_2)\big) g(t)\,dt + \ldots + \int_a^{c_{n-1}} \big(f(t) - f(d_{n-1})\big) g(t)\,dt \geq 0. \tag{1.8}$$

**(b)** Then

$$\int_{c_2}^{d_2} f(t)\,dt + \int_{c_3}^{d_3} f(t)\,dt + \ldots + \int_{c_{n-1}}^{d_{n-1}} f(t)\,dt + \int_{c_n}^{d_n} f(t)\,dt - r(c_n, d_n)$$

$$\leq (n-1)\int_a^b f(t)g(t)\,dt \tag{1.9}$$

holds where,

$$r(c_n, d_n) = \int_{d_2}^b \left(f(c_2) - f(t)\right)g(t)\,dt + \int_{d_3}^b \left(f(c_3) - f(t)\right)g(t)\,dt + \ldots$$

$$+\int_{d_{n-1}}^b \left(f(c_{n-1}) - f(t)\right)g(t)\,dt + \int_{d_n}^b \left(f(c_n) - f(t)\right)g(t)\,dt \geq 0. \tag{1.10}$$

## 2. Extensions of Cerone's Results for General Subintervals

Pečarić et al, [ 8 ] have introduced the following lemma in order to generalize Cerone's results for the function $f/k$ :

**Lemma 2.1** Let $k$ be a positive integrable function on $[a, b]$, and let $f, g, h : [a, b] \to \mathbb{R}$ be integrable functions on $[a,b]$. Further, let $[c,d] \subset [a,b]$ with $\int_c^d h(t)k(t)dt = \int_a^b g(t)k(t)dt$. Then, the following identities hold

$$\int_c^d f(t)h(t)\,dt - \int_a^b f(t)g(t)\,dt = \int_a^c \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\,dt$$

$$+\int_c^d \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right)k(t)\left(h(t) - g(t)\right)dt + \int_d^b \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\,dt, \tag{2.1}$$

and

$$\int_a^b f(t)g(t)\,dt - \int_c^d f(t)h(t)\,dt = \int_a^c \left(\frac{f(t)}{k(t)} - \frac{f(c)}{k(c)}\right)g(t)k(t)\,dt$$

$$+\int_c^d \left(\frac{f(c)}{k(c)}-\frac{f(t)}{k(t)}\right)k(t)\big(h(t)-g(t)\big)\,dt+\int_d^b \left(\frac{f(t)}{k(t)}-\frac{f(c)}{k(c)}\right)g(t)k(t)\,dt. \tag{2.2}$$

The following theorems have been established by Pečarić et al, [ 8 ] as a new generalizations of Cerone's result for the function $f/k$

**Theorem 2.1** Let $k$ be a positive integrable function on $[a,b]$, and let *f, g, h* $:[a,b]\to\mathbb{R}$ be integrable functions on $[a,b]$ such that $f/k$ is nonincreasing. Further, let $0\le g(t)\le h(t)$ and $\int_{c_i}^{d_i} h(t)k(t)\,dt=\int_a^b g(t)k(t)\,dt$, where $[c_i,d_i]\subset[a,b]$ for $i=1,2$ and $d_1\le d_2$.

**(a)** Then

$$\int_a^b f(t)g(t)\,dt \le \int_{c_1}^{d_1} f(t)h(t)\,dt + R(c_1,d_1) \tag{2.3}$$

holds, where

$$R(c_1,d_1)=\int_a^{c_1}\left(\frac{f(t)}{k(t)}-\frac{f(d_1)}{k(d_1)}\right)g(t)k(t)\,dt \ge 0. \tag{2.4}$$

**(b)** Then

$$\int_{c_2}^{d_2} f(t)h(t)\,dt - r(c_2,d_2) \le \int_a^b f(t)g(t)\,dt \tag{2.5}$$

holds, where

$$r(c_2,d_2)=\int_{d_2}^{b}\left(\frac{f(c_2)}{k(c_2)}-\frac{f(t)}{k(t)}\right)g(t)k(t)\,dt \ge 0. \tag{2.6}$$

If $f/k$ is a nondecreasing function, then the inequalities in (2.3), (2.4), (2.5) and (2.6) are reversed.

**Remarks 2.1**

**(a)** If we take $c_1 = a$ and $d_1 = a + \lambda$ and we take $k(t) = 1$ in Theorem 2.1 of part (a), we obtain a Mercer's generalization of the right-hand Steffensen's inequality(see [5, Theorem 3]).

**(b)** If we take $c_2 = b - \lambda$ and $d_2 = b$ and we take $k(t) = 1$ in Theorem 2.1 of part (b), we obtain a Mercer's generalization of the left-hand Steffensen's inequality which is obtained in [9] from a generalization given by Pečarić in [10] (see [9, Theorem 2.7]).

**(c)** If we take $h(t) = k(t) = 1$ in Theorem 2.1, then we obtain Theorem 1.2

The aim of this article is to introduce more extensions to the work of Pečarić et al,[7]. Let us first extend the results of Theorem 2.1 for the case $i = 1, 2, 3$.

**Theorem 2.2** Let $k$ be a positive integrable function on $[a, b]$, and let $f, g, h : [a, b] \to \mathbb{R}$ be integrable functions on $[a, b]$ such that $f/k$ is nonincreasing. Further, let $0 \le g(t) \le h(t)$ and $\int_{c_i}^{d_i} h(t)k(t)dt = \int_a^b g(t)k(t)dt$, where $[c_i, d_i] \subset [a, b]$ for $i = 1, 2, 3$ and $d_1 \le d_2 \le d_3$.

**(a)** Then

$$2\int_a^b f(t)g(t)\,dt \le \int_{c_1}^{d_1} f(t)h(t)\,dt + \int_{c_2}^{d_2} f(t)h(t)\,dt + R(c_2, d_2) \tag{2.7}$$

holds, where

$$R(c_2, d_2) = \int_a^{c_1} \left( \frac{f(t)}{k(t)} - \frac{f(d_1)}{k(d_1)} \right) g(t)k(t)\,dt$$

$$+\int_a^{c_2}\left(\frac{f(t)}{k(t)}-\frac{f(d_2)}{k(d_2)}\right)g(t)k(t)\,dt \ge 0. \qquad (2.8)$$

**(b)** Then

$$\int_{c_2}^{d_2} f(t)h(t)\,dt+\int_{c_3}^{d_3} f(t)h(t)\,dt-r(c_3,d_3)\le 2\int_a^b f(t)g(t)\,dt \qquad (2.9)$$

holds, where

$$r(c_3,d_3)=\int_{d_2}^{b}\left(\frac{f(c_2)}{k(c_2)}-\frac{f(t)}{k(t)}\right)g(t)k(t)\,dt$$

$$+\int_{d_3}^{b}\left(\frac{f(c_3)}{k(c_3)}-\frac{f(t)}{k(t)}\right)g(t)k(t)\,dt \ge 0. \qquad (2.10)$$

If $f/k$ is a nondecreasing function, then the inequalities in (2.7), (2.8), (2.9) and (2.10) are reversed.

**Proof of part (a):** Since $\frac{f(t)}{k(t)}$ is nonincreasing on $[c_1,d_1]$, then $\frac{f(t)}{k(t)}\ge\frac{f(d_1)}{k(d_1)}$, and $\frac{f(t)}{k(t)}$ is nonincreasing on $[c_2,d_2]$, then $\frac{f(t)}{k(t)}\ge\frac{f(d_2)}{k(d_2)}$. Similarly, $\frac{f(t)}{k(t)}$ is nonincreasing on $[d_1,b]$, then $\frac{f(t)}{k(t)}\le\frac{f(d_1)}{k(d_1)}$,and $\frac{f(t)}{k(t)}$ is nonincreasing on $[d_2,b]$, then $\frac{f(t)}{k(t)}\le\frac{f(d_2)}{k(d_2)}$.

Also since $0\le g(t)\le h(t)$, we get $0\ge -g(t)\ge -h(t)$, and then $0\le h(t)-g(t)\le h(t)$.

Now, let us introduce

$$L(c_1,d_1;a,b)=\int_{c_1}^{d_1} f(t)h(t)dt-\int_a^b f(t)g(t)dt,\quad a\le c_1\le d_1\le b$$

and

$$L(c_2,d_2;a,b)=\int_{c_2}^{d_2} f(t)\,h(t)\,dt-\int_a^b f(t)\,g(t)\,dt,\quad a\le c_2\le d_2\le b.$$

By using Lemma 2.1, we obtain

$$L(c_1,d_1;a,b)+L(c_2,d_2;a,b)+\int_a^{c_1}\left(\frac{f(t)}{k(t)}-\frac{f(d_1)}{k(d_1)}\right)g(t)\,k(t)\,dt$$

$$+\int_a^{c_2}\left(\frac{f(t)}{k(t)}-\frac{f(d_2)}{k(d_2)}\right)g(t)\,k(t)\,dt$$

$$=\int_{c_1}^{d_1}\left(\frac{f(t)}{k(t)}-\frac{f(d_1)}{k(d_1)}\right)k(t)\big(h(t)-g(t)\big)\,dt+\int_{c_2}^{d_2}\left(\frac{f(t)}{k(t)}-\frac{f(d_2)}{k(d_2)}\right)k(t)\big(h(t)-g(t)\big)\,dt$$

$$+\int_{d_1}^{b}\left(\frac{f(d_1)}{k(d_1)}-\frac{f(t)}{k(t)}\right)g(t)\,k(t)\,dt+\int_{d_2}^{b}\left(\frac{f(d_2)}{k(d_2)}-\frac{f(t)}{k(t)}\right)g(t)\,k(t)\,dt\ge 0.$$

Hence

$$\int_{c_1}^{d_1} f(t)h(t)\,dt-\int_a^b f(t)g(t)\,dt+\int_{c_2}^{d_2} f(t)h(t)\,dt-\int_a^b f(t)g(t)\,dt$$

$$+\int_a^{c_1}\left(\frac{f(t)}{k(t)}-\frac{f(d_1)}{k(d_1)}\right)g(t)\,k(t)\,dt+\int_a^{c_2}\left(\frac{f(t)}{k(t)}-\frac{f(d_2)}{k(d_2)}\right)g(t)\,k(t)\,dt\ge 0$$

and thus (2.7) is valid. The term $R(c_2,d_2)$ is nonnegative since $f/k$ is nonincreasing and *g* is nonnegative, *k* is positive ■

**Proof of part (b):** Since $\dfrac{f(t)}{k(t)}$ is nonincreasing on $[a,c_2]$, for $t\le c_2$, then $\dfrac{f(t)}{k(t)}\ge\dfrac{f(c_2)}{k(c_2)}$,

and $\dfrac{f(t)}{k(t)}$ is nonincreasing on $[a,c_3]$, for $t\le c_3$, then

$\dfrac{f(t)}{k(t)}\ge\dfrac{f(c_3)}{k(c_3)}$. Similarly, $\dfrac{f(t)}{k(t)}$ is nonincreasing on $[c_2, d_2]$, for $t \geq c_2$, then $\frac{f(t)}{k(t)} \leq \frac{f(c_2)}{k(c_2)}$, and

$\frac{f(t)}{k(t)}$ is nonincreasing on $[c_3, d_3]$, for $t \geq c_3$, then $\frac{f(t)}{k(t)} \leq \frac{f(c_3)}{k(c_3)}$.

Also since $0 \leq g(t) \leq h(t)$, we get $0 \geq -g(t) \geq -h(t)$, then $0 \leq h(t) - g(t) \leq h(t)$.

Now, using Lemma 2.1, we obtain

$$-L(c_2, d_2; a, b) - L(c_3, d_3; a, b) + \int_{d_2}^{b} \left( \frac{f(c_2)}{k(c_2)} - \frac{f(t)}{k(t)} \right) g(t) k(t)\, dt$$

$$+ \int_{d_3}^{b} \left( \frac{f(c_3)}{k(c_3)} - \frac{f(t)}{k(t)} \right) g(t) k(t)\, dt$$

$$= \int_{a}^{c_2} \left( \frac{f(t)}{k(t)} - \frac{f(c_2)}{k(c_2)} \right) g(t) k(t)\, dt + \int_{a}^{c_3} \left( \frac{f(t)}{k(t)} - \frac{f(c_3)}{k(c_3)} \right) g(t) k(t)\, dt$$

$$+ \int_{c_2}^{d_2} \left( \frac{f(c_2)}{k(c_2)} - \frac{f(t)}{k(t)} \right) k(t) \big( h(t) - g(t) \big)\, dt + \int_{c_3}^{d_3} \left( \frac{f(c_3)}{k(c_3)} - \frac{f(t)}{k(t)} \right) k(t) \big( h(t) - g(t) \big)\, dt \geq 0.$$

Hence

$$\int_{a}^{b} f(t) g(t)\, dt - \int_{c_2}^{d_2} f(t) h(t)\, dt + \int_{a}^{b} f(t) g(t)\, dt - \int_{c_3}^{d_3} f(t) h(t)\, dt$$

$$+ \int_{d_2}^{b} \left( \frac{f(c_2)}{k(c_2)} - \frac{f(t)}{k(t)} \right) g(t) k(t)\, dt + \int_{d_3}^{b} \left( \frac{f(c_3)}{k(c_3)} - \frac{f(t)}{k(t)} \right) g(t) k(t)\, dt \geq 0$$

giving (2.9) is valid. The term $r(c_3, d_3)$ is nonnegative since $f/k$ is nonincreasing and *g* is nonnegative, *k* is positive ■

The following theorem provides more extension of Theorem 2.2 to the case $i = 1, 2, \ldots, n$.

**Theorem 2.3** Let $k$ be a positive integrable function on $[a,b]$, and let $f, g, h : [a,b] \to \mathbb{R}$ be integrable functions on $[a,b]$ such that $f/k$ is nonincreasing. Further, let $0 \le g(t) \le h(t)$ and $\int_{c_i}^{d_i} h(t)k(t)\,dt = \int_a^b g(t)k(t)\,dt$, where $[c_i, d_i] \subset [a,b]$ for $i = 1, 2, \ldots, n$ and $d_1 \le d_2 \le \ldots \le d_n$, $n = 2, 3, 4, \ldots$ .

**(a)** Then

$$(n-1)\int_a^b f(t)g(t)\,dt \le \int_{c_1}^{d_1} f(t)h(t)\,dt + \int_{c_2}^{d_2} f(t)h(t)\,dt + \ldots + \\ + \int_{c_{n-1}}^{d_{n-1}} f(t)h(t)\,dt + R(c_{n-1}, d_{n-1}) \quad (2.11)$$

holds, where

$$R(c_{n-1}, d_{n-1}) = \int_a^{c_1} \left( \frac{f(t)}{k(t)} - \frac{f(d_1)}{k(d_1)} \right) g(t)k(t)\,dt + \int_a^{c_2} \left( \frac{f(t)}{k(t)} - \frac{f(d_2)}{k(d_2)} \right) g(t)k(t)\,dt + \ldots \\ + \int_a^{c_{n-1}} \left( \frac{f(t)}{k(t)} - \frac{f(d_{n-1})}{k(d_{n-1})} \right) g(t)k(t)\,dt \ge 0. \quad (2.12)$$

**(b)** Then

$$\int_{c_2}^{d_2} f(t)h(t)\,dt + \int_{c_3}^{d_3} f(t)h(t)\,dt + \ldots + \int_{c_{n-1}}^{d_{n-1}} f(t)h(t)\,dt$$

$$+ \int_{c_n}^{d_n} f(t)h(t)\,dt - r(c_n, d_n) \le (n-1)\int_a^b f(t)g(t)\,dt \quad (2.13)$$

holds, where

$$r(c_n, d_n) = \int_{d_2}^{b} \left( \frac{f(c_2)}{k(c_2)} - \frac{f(t)}{k(t)} \right) g(t)k(t)\,dt + \int_{d_3}^{b} \left( \frac{f(c_3)}{k(c_3)} - \frac{f(t)}{k(t)} \right) g(t)k(t)\,dt + \ldots$$

$$+\int_{d_{n-1}}^{b}\left(\frac{f(c_{n-1})}{k(c_{n-1})}-\frac{f(t)}{k(t)}\right)g(t)k(t)\,dt+\int_{d_n}^{b}\left(\frac{f(c_n)}{k(c_n)}-\frac{f(t)}{k(t)}\right)g(t)k(t)\,dt\ge 0. \tag{2.14}$$

If $f/k$ is a nondecreasing function, then the inequalities in (2.11), (2.12), (2.13) and (2.14) are reversed.

**Proof of part (a):** Since $\frac{f(t)}{k(t)}$ is nonincreasing on $[c_1, d_1]$ for $t \le d_1$, on $[c_2, d_2]$ for $t \le d_2$, , . . . , on $[c_{n-1}, d_{n-1}]$ for $t \le d_{n-1}$ respectively, then

$$\frac{f(t)}{k(t)}\ge\frac{f(d_1)}{k(d_1)},\ \frac{f(t)}{k(t)}\ge\frac{f(d_2)}{k(d_2)},\ \ldots,\ \frac{f(t)}{k(t)}\ge\frac{f(d_{n-1})}{k(d_{n-1})}$$

respectively. Similarly, $\frac{f(t)}{k(t)}$ is nonincreasing on $[d_1, b]$ for $t \ge d_1$, on $[d_2, b]$ for $t \ge d_2$, . . . , on $[d_{n-1}, b]$ for $t \ge d_{n-1}$, respectively, then

$$\frac{f(t)}{k(t)}\le\frac{f(d_1)}{k(d_1)},\frac{f(t)}{k(t)}\le\frac{f(d_2)}{k(d_2)},\ldots,\frac{f(t)}{k(t)}\le\frac{f(d_{n-1})}{k(d_{n-1})}$$

respectively. Also since $0 \le g(t) \le h(t)$, we get $0 \ge -g(t) \ge -h(t)$, then $0 \le h(t)-g(t) \le h(t)$, and from Lemma 2.1, we obtain

$$L(c_1,d_1;a,b)+L(c_2,d_2;a,b)+\ldots+L(c_{n-1},d_{n-1};a,b)+\int_a^{c_1}\left(\frac{f(t)}{k(t)}-\frac{f(d_1)}{k(d_1)}\right)g(t)k(t)\,dt$$

$$+\int_a^{c_2}\left(\frac{f(t)}{k(t)}-\frac{f(d_2)}{k(d_2)}\right)g(t)k(t)\,dt+\ldots+\int_a^{c_{n-1}}\left(\frac{f(t)}{k(t)}-\frac{f(d_{n-1})}{k(d_{n-1})}\right)g(t)k(t)\,dt$$

$$= \int_{c_1}^{d_1} \left( \frac{f(t)}{k(t)} - \frac{f(d_1)}{k(d_1)} \right) k(t)\big(h(t)-g(t)\big)\, dt + \int_{c_2}^{d_2} \left( \frac{f(t)}{k(t)} - \frac{f(d_2)}{k(d_2)} \right) k(t)\big(h(t)-g(t)\big)\, dt$$

$$+ \ldots + \int_{c_{n-1}}^{d_{n-1}} \left( \frac{f(t)}{k(t)} - \frac{f(d_{n-1})}{k(d_{n-1})} \right) k(t)\big(h(t)-g(t)\big)\, dt + \int_{d_1}^{b} \left( \frac{f(d_1)}{k(d_1)} - \frac{f(t)}{k(t)} \right) g(t)k(t)\, dt$$

$$+ \int_{d_2}^{b} \left( \frac{f(d_2)}{k(d_2)} - \frac{f(t)}{k(t)} \right) g(t)k(t)\, dt + \ldots + \int_{d_{n-1}}^{b} \left( \frac{f(d_{n-1})}{k(d_{n-1})} - \frac{f(t)}{k(t)} \right) g(t)k(t)\, dt \geq 0$$

Hence

$$\int_{c_1}^{d_1} f(t)h(t)\, dt + \int_{c_2}^{d_2} f(t)h(t)\, dt + \ldots + \int_{c_{n-1}}^{d_{n-1}} f(t)h(t)\, dt - \int_{a}^{b} f(t)g(t)\, dt$$

$$- \int_{a}^{b} f(t)g(t)\, dt - \ldots - \int_{a}^{b} f(t)g(t)\, dt + \int_{a}^{c_1} \left( \frac{f(t)}{k(t)} - \frac{f(d_1)}{k(d_1)} \right) g(t)k(t)\, dt$$

$$+ \int_{a}^{c_2} \left( \frac{f(t)}{k(t)} - \frac{f(d_2)}{k(d_2)} \right) g(t)k(t)\, dt + \ldots + \int_{a}^{c_{n-1}} \left( \frac{f(t)}{k(t)} - \frac{f(d_{n-1})}{k(d_{n-1})} \right) g(t)k(t)\, dt \geq 0$$

and thus (2.11) is valid. The term $R(c_{n-1}, d_{n-1})$ is nonnegative since $f/k$ is nonincreasing and *g* is nonnegative, *k* is positive ■

**Proof of part (b):** Since $\frac{f(t)}{k(t)}$ is nonincreasing on $[a, c_2]$ for $t \leq c_2$, on $[a, c_3]$ for $t \leq c_3$, $\frac{f(t)}{k(t)} \geq \frac{f(c_n)}{k(c_n)}$ respectively. Similarly, $\frac{f(t)}{k(t)}$ is nonincreasing on $[c_2, d_2]$ for $t \geq c_2$, on $[c_3, d_3]$ for $t \geq c_3$,

. . . , on $[c_n, d_n]$ for $t \geq c_n$ , respectively, then $\frac{f(t)}{k(t)} \leq \frac{f(c_2)}{k(c_2)}$,

$\frac{f(t)}{k(t)} \leq \frac{f(c_3)}{k(c_3)}, \ldots, \frac{f(t)}{k(t)} \leq \frac{f(c_n)}{k(c_n)}$ ,

respectively. Also since $0 \leq g(t) \leq h(t)$, we get $0 \geq -g(t) \geq -h(t)$, then $0 \leq h(t) - g(t) \leq h(t)$, and from Lemma 2.1, we obtain

$$-L(c_2, d_2; a, b) - L(c_3, d_3; a, b) - \ldots - L(c_n, d_n; a, b) + \int_{d_2}^{b} \left( \frac{f(c_2)}{k(c_2)} - \frac{f(t)}{k(t)} \right) g(t) k(t)\, dt$$

$$+ \int_{d_3}^{b} \left( \frac{f(c_3)}{k(c_3)} - \frac{f(t)}{k(t)} \right) g(t) k(t)\, dt + \ldots + \int_{d_n}^{b} \left( \frac{f(c_n)}{k(c_n)} - \frac{f(t)}{k(t)} \right) g(t) k(t)\, dt$$

$$= \int_{a}^{c_2} \left( \frac{f(t)}{k(t)} - \frac{f(c_2)}{k(c_2)} \right) g(t) k(t)\, dt + \int_{a}^{c_3} \left( \frac{f(t)}{k(t)} - \frac{f(c_3)}{k(c_3)} \right) g(t) k(t)\, dt + \ldots$$

$$+ \int_{a}^{c_n} \left( \frac{f(t)}{k(t)} - \frac{f(c_n)}{k(c_n)} \right) g(t) k(t)\, dt + \int_{c_2}^{d_2} \left( \frac{f(c_2)}{k(c_2)} - \frac{f(t)}{k(t)} \right) k(t) \big( h(t) - g(t) \big)\, dt$$

$$+ \int_{c_3}^{d_3} \left( \frac{f(c_3)}{k(c_3)} - \frac{f(t)}{k(t)} \right) k(t) \big( h(t) - g(t) \big)\, dt + \ldots$$

$$+ \int_{c_n}^{d_n} \left( \frac{f(c_n)}{k(c_n)} - \frac{f(t)}{k(t)} \right) k(t) \big( h(t) - g(t) \big)\, dt \geq 0.$$

Hence

$$\int_a^b f(t) g(t)\, dt + \int_a^b f(t) g(t)\, dt + \ldots + \int_a^b f(t) g(t)\, dt - \int_{c_2}^{d_2} f(t) h(t)\, dt$$

$$- \int_{c_3}^{d_3} f(t) h(t)\, dt - \ldots - \int_{c_n}^{d_n} f(t) h(t)\, dt + \int_{d_2}^{b} \left( \frac{f(c_2)}{k(c_2)} - \frac{f(t)}{k(t)} \right) g(t) k(t)\, dt$$

$$+ \int_{d_3}^{b} \left( \frac{f(c_3)}{k(c_3)} - \frac{f(t)}{k(t)} \right) g(t) k(t)\, dt + \ldots + \int_{d_n}^{b} \left( \frac{f(c_n)}{k(c_n)} - \frac{f(t)}{k(t)} \right) g(t) k(t)\, dt \geq 0 \tag{2.15}$$

giving (2.12) is valid. The term $r(c_n, d_n)$ is nonnegative since $f/k$ is nonincreasing and $g$ is nonnegative, $k$ is positive ■

**Remark 2.2** Theorem 1.3 is a consequence of Theorem 2.3 by taking $h(t) = k(t) = 1$.

**Conclusion :**

In this article, based on the work of Cerone [3] and Pecaric et al [7], a new Steffensen type inequality has been researched . Our results are more general than those in [7]. We expect that the results may attract other researchers to employ them in their future research.

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