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Inference on the Transmuted Burr Type XII distribution

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Abstract:

Burr XII distribution is one of the most important and most widely used distributions for lifetime as well as the wealth data analysis and modeling. Burr Type XII is a member of a system of continuous distributions introduced by Burr (1942). Adding parameters to a well-established distribution is a time honored device for obtaining more flexible new families of distributions.

In this paper, we derived a new model as a generalization for the Burr XII distribution, namely; the transmuted Burr XII distribution (TBXII). We discussed the basic characteristics of the TBXII such as the reliability, hazard rate function , rth moment, expectation, variance, median, moment generating function, and the probability density function (pdf) of the rth order statistic. We also estimated the parameters using the method of moments, Maximum Likelihood Estimation method, Quantile Matching method and maximum goodness-of-fit Method . We finally applied the TBXII distribution to a simulated dataset and conducted a simulation study to compare methods of the parameters of the TBXII distribution.

As a result, we found that the TBXII distribution have very interesting distributional properties and the estimates of the parameters of the distribution are very good. The TBXII distribution is a very flexible one and can be particularly applied to heavy tailed datasets such as lifetime data in health and finance.

1. Introduction

Burr XII distribution is one of the most important and most widely used distributions for lifetime as well as the wealth data analysis and modeling. Burr Type XII is a member of a system of continuous distributions introduced by Burr (1942). Adding parameters to a well-established distribution is a time honored device for obtaining more flexible new families of distributions. Shaw and Bukley (2007) pioneered an interesting method of adding a new parameter to an existing distribution that would offer more distributional flexibility. They used the quadratic rank transmutation map (QRTM) in order to generate a flexible family of distributions.

In this paper, we derived a new model as a generalization for the *Burr XII* distribution, namely; the transmuted Burr XII distribution (*TBXII*). We discussed the basic characteristics of the TBXII such as the reliability, hazard rate function , r^{th} moment, expectation, variance, median, moment generating function, and the probability density function (pdf) of the r^{th} order statistic. We also estimated the parameters using the method of moments, Maximum Likelihood Estimation method, Quantile Matching method and maximum goodness-of-fit Method . We finally applied the *TBXII* distribution to a simulated dataset and conducted a simulation study to compare methods of the parameters of the *TBXII* distribution.

As a result, we found that the *TBXII* distribution have very interesting distributional properties and the estimates of the parameters of the distribution are very good. The *TBXII* distribution is a very flexible one and can be particularly applied to heavy tailed datasets such as lifetime data in health and finance.

In section 2, we introduced the TBXII distribution. We discussed its reliability properties in section 3. In section 4 we discussed the basic characteristics of the TBXII such as r^{th} moment, expectation, variance, median and moment generating function . The probability density function (pdf) of the r^{th} order statistic is derived in section 5 . In section 6, we presented examples of related distributions . Section 7 dealt with estimation of the parameters of TBXII distribution . While the four estimation methods were compared in section 8 . In section 9 , we conducted a simulation study and also presented a real data analysis in section 10 . Finally a conclusion is given in section 11 .

2. Transmuted Burr XII Distribution and its Properties

The probability density function (pdf) of the two parameter *Burr XII* distribution is defined as

$$g\left(x;k,c\right) \;=\; ck\; x^{c-1}\left(1+x^{c}\right)^{-(k+1)} \qquad ; \quad x>0 \qquad (1)$$

and its corresponding cumulative distribution function (cdf) is given by

$$G\left(x;k,c\right)= 1-\left(1+x^{c}\right)^{-k} \qquad ; \quad x>0 \qquad (2)$$

where k , c> 0 are the shape parameters.

A random variable X is said to have transmuted distribution if its cumulative distribution function (cdf) is given by

$$F(x) \;=(1+\lambda)G(x)-\lambda G^{2}\left(x\right) \qquad ; \quad \left|\lambda\right|\leq 1 \qquad (3)$$

where G(x) is the cdf of the base distribution . If $\lambda=0$, we have the distribution of the base random variable. From (3) we have :

$$f\,(x) \;=g(x)\left[1+\lambda-2\lambda G(x)\right] \qquad ; \quad \left|\lambda\right|\leq 1 \qquad (4)$$

where f(x) and g(x) are the corresponding probability density function (pdf) of F(x) and G(x) , respectively .

Let X be a random variable having transmuted Burr XII (TBXII) distribution ,and then the cumulative distribution function (cdf) have the form given as follows :

$$F(x)=1+\left[(\lambda-1)\left(1+x^{c}\right)^{-k}-\lambda\left(1+x^{c}\right)^{-2k}\right]; \qquad |\lambda|\le 1, x>0, c>0, k>0 \quad , \qquad (5)$$

where c > 0 and k > 0 are the shape parameters and λ is the scale parameter of the distribution . The probability density function (pdf) of $\text{TBXII}(\text{c,k},\lambda)$ distribution have the form :

$$f(x) \;=ckx^{c-1}\left(1+x^{c}\right)^{-k-1}\left[1-\lambda+2\lambda\left(1+x^{c}\right)^{-k}\right]; \qquad |\lambda|\le 1, x>0 \qquad (6)$$

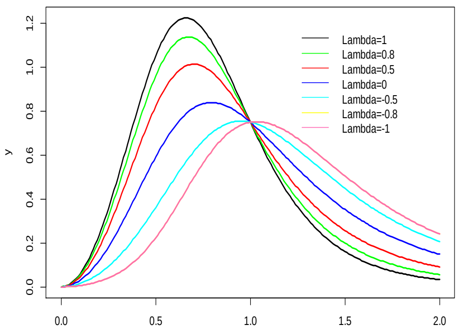

Fig.(1.a): Shape of Density Function when c=3,k=1 and different values of Lambda

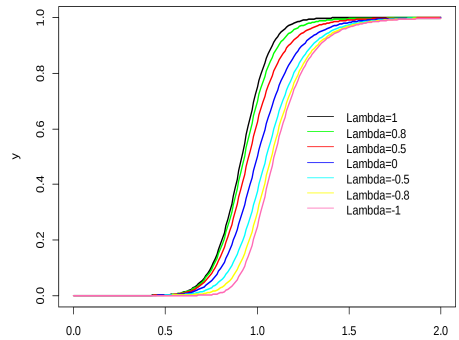

Fig.(1.b): Shape of CDF when c=1,k=1 and different values of Lambda

Figures (1.a) and (1.b) illustrate the graphs of various shapes of the density and the cumulative distribution functions of $\text{TBXII}(\text{c,k},\lambda)$ respectively with different values of λ and fixed values of c and k . Whenever the value of λ increased, the flattening form of the probability density function increased under the fixed values of c and k. Moreover, whenever the values of λ decrease, the cumulative distribution function approaches the x- axis .

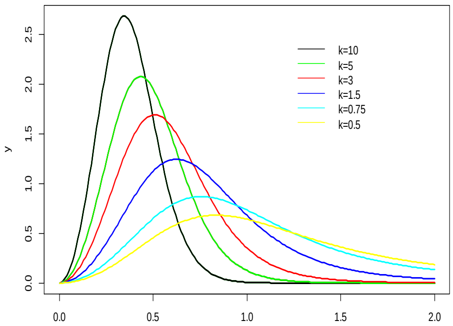

Fig.(2.a): Shape of Density Function when c=3,lambda=0.5and different values of k

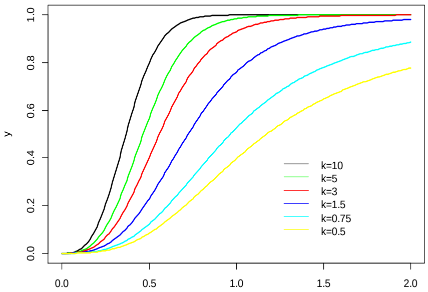

Fig.(2.b): Shape of CDF when c=3 , lambda=0.5 and different values of k

Figures (2.a) and (2.b) illustrates graphs of various shapes of the density and the cumulative distribution functions of $\text{TBXII}(c,k,\lambda)$ respectively with different values of k and fixed value of c and λ. Whenever the values of k increased, the flattening form of the probability density function decreased under the fixed values of c and λ. Moreover, whenever the value of k increase, the cumulative distribution function becomes closer to the y-axis .

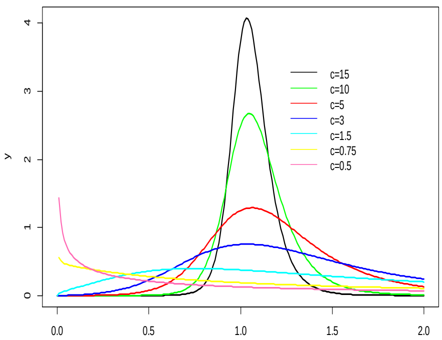

Fig.(3.a): Shape of Density Function when k=1,lambda=0.8and different values of c

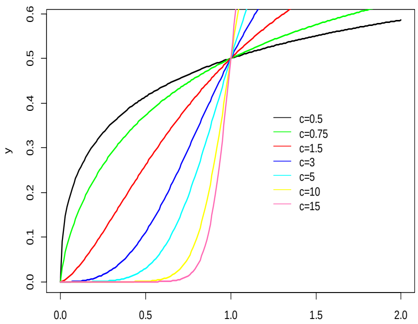

Fig.(3.b): Shape of CDF when k=0.5 , lambda=1 and different values of c

Figures (3.a) and (3.b) illustrate the graphs of various shapes of the density and the cumulative distribution functions of $\mathrm{TBXII}(\mathrm{c,k},\lambda)$ respectively with different values of *c* and fixed value of *k* and λ. Whenever the values of *c* increased, the flattening form of the probability density function will be decreased under the fixed values of *k* and λ. Moreover, whenever the values of *c* increase, the cumulative distribution function approaching to the x-axis if $\mathrm{x} \in [0,1]$, but if x > 1 then when the values of *c* increase, the cumulative distribution function goes away from x-axis.

3. Reliability analysis of the TBXII distribution

Let X be a random variable having the $\mathrm{TBXII}(\mathrm{c,k},\lambda)$ distribution with probability density function $f_{TBXII}(x)$ and cumulative distribution function $F_{TBXII}(x)$. The reliability function of the TBXII distribution is defined as the following

$$R_{TBXII}(x) = 1 - F_{TBXII}(x)$$

$$\begin{aligned} R_{TBXII}(X) &= 1 - \left(\left[1+\lambda\left(1+x^{c}\right)^{-k}\right]\left[1-\left(1+x^{c}\right)^{-k}\right]\right) \\ &= \left(1+x^{c}\right)^{-k}\left[1-\lambda+\lambda\left(1+x^{c}\right)^{-k}\right] \end{aligned} \tag{7}$$

The hazard rate functions of the TBXII distribution is defined as the following

$$H_{TBXII}(x) = \frac{f_{TBXII}(x)}{R_{TBXII}(x)}$$

$$H_{TBXII}(x) = \frac{ckx^{c-1}\left(1+x^{c}\right)^{-k-1}\left[1-\lambda+2\lambda\left(1+x^{c}\right)^{-k}\right]}{1-\left(\left[1+\lambda\left(1+x^{c}\right)^{-k}\right]\left[1-\left(1+x^{c}\right)^{-k}\right]\right)}$$

$$= \frac{ckx^{c-1}\left[1-\lambda+2\lambda\left(1+x^{c}\right)^{-k}\right]}{\left(1+x^{c}\right)\left[1-\lambda+\lambda\left(1+x^{c}\right)^{-k}\right]} \qquad (8)$$

4. The r^{th} Moment for the Transmuted Burr XII Distribution

The r^{th} moment of the Transmuted Burr XII(c,k,λ) distribution is

$$E\left(x^{r}\right)=\mu_{r}'=\int_{0}^{\infty} x^{r} ckx^{c-1}\left(1+x^{c}\right)^{-k-1}\left[1-\lambda+2\lambda\left(1+x^{c}\right)^{-k}\right]dx$$

$$=c\,k\int_{0}^{1}\left(\frac{1-u}{u}\right)^{\frac{r}{c}}\left(\frac{1-u}{u}\right)^{\frac{-1}{c}}\left(\frac{1-u}{u}\right)u^{k+1}\left[1-\lambda+2\lambda u^{k}\right]\frac{1}{c}(1-u)^{\frac{1}{c}-1}u^{-\left(1+\frac{1}{c}\right)}du$$

$$= k\int_{0}^{1}(1-\mathrm{u})^{\frac{r}{c}}u^{k-\frac{r}{c}-1}\left[1-\lambda+2\lambda u^{k}\right]du$$

$$= k\left(1-\lambda\right)\int_{0}^{1}(1-\mathrm{u})^{\left(\frac{r}{c}+1\right)-1}u^{\left(k-\frac{r}{c}\right)-1}du+2\lambda k\int_{0}^{1}\left(1-u\right)^{\left(\frac{r}{c}+1\right)-1}u^{\left(2k-\frac{r}{c}\right)-1}du$$

$$= k\left(1-\lambda\right)\beta\left(k-\frac{r}{c},\frac{r}{c}+1\right)+2\lambda k\,\beta\left(2k-\frac{r}{c},\frac{r}{c}+1\right),\quad r<c \qquad (9)$$

4.1. The expectation of the TBXII distribution

Let X be a random variables having the TBXII(c,k,λ) distribution, then from equation (5) for the $r=1$,

$$\mu = E(X) = \frac{1-\lambda}{c}\ \beta\left(k-\frac{1}{c},\frac{1}{c}\right)+\frac{\lambda}{c}\beta\left(2k-\frac{1}{c},\frac{1}{c}\right)$$

$$= k(1-\lambda)\beta\left(k-\frac{1}{c},\frac{1}{c}+1\right)+2k\lambda\beta\left(2k-\frac{1}{c},\frac{1}{c}+1\right),\quad c>1 \qquad (10)$$

4.2. The variance of the TBXII distribution

Let X be a random variables having the $Burr\ XII(c,k)$ distribution, then from equation (5) for the $r=2$, we have:

$$E\left(X^2\right) = k(1-\lambda)\ \beta\left(k-\frac{2}{c},\frac{2}{c}+1\right)+2k\lambda\beta\left(2k-\frac{2}{c},\frac{2}{c}+1\right)$$

Therefore ,

$$\sigma^2 = var(X) = k(1-\lambda)\beta\left(k-\frac{2}{c},\frac{2}{c}+1\right)+2k\lambda\beta\left(2k-\frac{2}{c},\frac{2}{c}+1\right)-$$

$$\left[k(1-\lambda)\beta\left(k-\frac{1}{c},\frac{1}{c}+1\right)+2k\lambda\beta\left(2k-\frac{1}{c},\frac{1}{c}+1\right)\right]^2, c>2 \qquad (11)$$

4.3. The median of the TBXII distribution

The median of the $\mathrm{TBXII}(c,k,\lambda)$ distribution is the value x at which $P(X \le x) = F(X) = 0.5$ **.** Thus,

$$\left[1+\lambda\left(1+x^c\right)^{-k}\right]\left[1-\left(1+x^c\right)^{-k}\right]=0.5$$

$$1-\left(1+x^c\right)^{-k}+\lambda\left(1+x^c\right)^{-k}-\lambda\left(1+x^c\right)^{-2k}=0.5$$

Let $y=\left(1+x^c\right)^{-k}$

$$\Rightarrow \lambda y^2+(1-\lambda)y-0.5=0$$

then $\quad y = \dfrac{(\lambda-1) \pm \sqrt{(1-\lambda)^2 - 4(-0.5)\lambda}}{2\lambda}$

$$= \frac{(\lambda-1) \pm \sqrt{1+\lambda^2}}{2\lambda}$$

then , $\quad \left(1+x^c\right)^{-k} = \dfrac{(\lambda-1) \pm \sqrt{1+\lambda^2}}{2\lambda}$

$$x^c = \left[\frac{(\lambda-1) \pm \sqrt{1+\lambda^2}}{2\lambda}\right]^{\frac{-1}{k}} - 1$$

$$\therefore\ x = \left[\left[\frac{(\lambda-1) \pm \sqrt{1+\lambda^2}}{2\lambda}\right]^{\frac{-1}{k}} - 1\right]^{\frac{1}{c}}$$

(12)

4.5. The moment generating function of the TBXII distribution

For a random variable X which follows a TBXII$\left(\text{c,k,}\lambda\right)$ distribution, the moment generating function (*mgf*) is:-

$$\mathrm{M}_x\ (t)\ =\ E\left(e^{tx}\right) = \int_0^{\infty} e^{tx} f\ (x)dx$$

$$= \int_0^{\infty} e^{tx} ck\ x^{c-1}\left(1+x^c\right)^{-k-1}\left[1-\lambda+2\lambda\left(1+x^c\right)^{-k}\right]dx$$

Using the negative binomial expansion for the following expression

$$\left(1+x^c\right)^{-k} = \sum_{r=0}^{\infty}(-1)^r\binom{k+r-1}{r}\left(x^c\right)^r = \sum_{r=0}^{\infty}\binom{-k}{r}\left(x^c\right)^r$$

$$\left(1+x^c\right)^{-k-1} = \sum_{m=0}^{\infty}(-1)^r\binom{k+m}{m}\left(x^c\right)^m = \sum_{m=0}^{\infty}\binom{-k-1}{m}\left(x^c\right)^m,$$

then

$$M(X)=E(e^{tx})=\int_0^\infty e^{tx}ckx^{c-1}\sum_{m=0}^{\infty}\binom{-k-1}{m}\left(x^c\right)^m\left[1-\lambda+2\lambda\sum_{r=0}^{\infty}\binom{-k}{r}\left(x^c\right)^r\right]dx$$

$$=\int_0^\infty\left[ck\binom{-k-1}{m}\sum_{m=0}^{\infty}e^{tx}x^{c-1}\left(x^c\right)^m-\lambda ck\binom{-k-1}{m}\sum_{m=0}^{\infty}e^{tx}x^{c-1}\left(x^c\right)^m\right.$$
$$\left.+2\lambda ck\binom{-k-1}{m}\binom{-k}{r}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}e^{tx}x^{c-1}\left(x^c\right)^m\left(x^c\right)^r\right]dx$$

$$=ck\sum_{m=0}^{\infty}\binom{-k-1}{m}\int_0^\infty e^{tx}x^{c(m+1)-1}dx-\lambda ck\sum_{m=0}^{\infty}\binom{-k-1}{m}\int_0^\infty e^{tx}x^{c(m+1)-1}dx+$$
$$2\lambda ck\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\binom{-k-1}{m}\binom{-k}{r}\int_0^\infty e^{tx}x^{c(m+r+1)-1}dx$$

$$=\left[ck\sum_{m=0}^{\infty}\binom{-k-1}{m}-\lambda ck\sum_{m=0}^{\infty}\binom{-k-1}{m}\right]\left(\frac{-1}{t}\right)^{c(m+1)}+$$
$$2\lambda ck\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\binom{-k-1}{m}\binom{-k}{r}\left(\frac{-1}{t}\right)^{c(m+r+1)} \tag{13}$$

5. The pdf of the r^{th} order statistics of the TBXII distribution

Let $X_1, X_2, \ldots, X_n$ be n independently continuous random variables from the $TBXII$ (c,k,λ) and let $X_{1:n}\le X_{2:n}\le\ldots\le X_{n:n}$ be the corresponding order statistics.Let $f_{X_{(r:n)}}(x)$ denotes the probability density function of the r^{th} order statistics $X_{(r:n)}$. The probability density function of $X_{(r:n)}$ is given by Arnold et al. (1992) as:-

$$f_{X_{(r:n)}}(X)=\frac{1}{\beta(r,n-r+1)}f(x)\left(F(x)\right)^{r-1}\left(1-F(x)\right)^{n-r}$$

$$f_{X_{(r:n)}}(X)=\frac{1}{\beta(r,n-r+1)}ck\,x^{c-1}\left(1+x^{c}\right)^{-k-1}\left[1-\lambda+2\lambda\left(1+x^{c}\right)^{-k}\right]$$
$$\left\{\left[1+\lambda\left(1+x^{c}\right)^{-k}\right]\left[1-\left(1+x^{c}\right)^{-k}\right]\right\}^{r-1}\left\{1-\left[1+\lambda\left(1+x^{c}\right)^{-k}\right]\left[1-\left(1+x^{c}\right)^{-k}\right]\right\}^{n-r} \quad (14)$$

Let, $X_{1:n}=\min\{X_1,X_2,....,X_n\}$, and $X_{n:n}=\max\{X_1,X_2,....,X_n\}$ with pdfs $f_{X_{(1:n)}}(x)$ and $f_{X_{(n:n)}}(x)$ defined as follow

$$f_{X_{(1:n)}}(X)=nf(X)\left[1-F(X)\right]^{n-1} \text{ and } f_{X_{(n:n)}}(X)=nf(X)\left[F(X)\right]^{n-1}$$

As a special case of (14) , the pdf of the minimum and maximum order statistics, respectively, are

$$f_{X_{(1:n)}}(X)=n\,ck\,x^{c-1}\left(1+x^{c}\right)^{-k-1}\left[1-\lambda+2\lambda\left(1+x^{c}\right)^{-k}\right]\left\{1-\left[1+\lambda\left(1+x^{c}\right)^{-k}\right]\left[1-\left(1+x^{c}\right)^{-k}\right]\right\}^{n-1}$$

$$f_{X_{(n:n)}}(X)=n\,ck\,x^{c-1}\left(1+x^{c}\right)^{-k-1}\left[1-\lambda+2\lambda\left(1+x^{c}\right)^{-k}\right]\left\{\left[1+\lambda\left(1+x^{c}\right)^{-k}\right]\left[1-\left(1+x^{c}\right)^{-k}\right]\right\}^{n-1} \quad (15)$$

6. Related Distributions

Some distributions arise as special case of the TBXII(c,k,λ) distribution :

- When $\lambda=0$, we obtain a two- parameter Burr XII distribution with the pdf as
  $f(X)=ckx^{c-1}\left(1+x^{c}\right)^{-k-1}$
- When $k=1$, it becomes a two- parameter transmuted log logistic distribution with pdf as $f(X)=ckx^{c-1}\left(1+x^{c}\right)^{-k-1}$

- When $\lambda = 0$ *and* $k = 1$, it reduces to a one- parameter log logistic distribution with pdf as $f(X) = \dfrac{cx^{c-1}}{\left(1+x^{c}\right)^{2}}$
- When $c = 1$, it becomes a two- parameter transmuted lomax distribution with pdf

$$f(X) = k\left(1+x\right)^{-k-1}\left[1-\lambda+2\lambda\left(1+x\right)^{-1}\right]$$

- When $\lambda = 0$ *and* $c = 1$, it reduces to one- parameter lomax distribution with pdf as $f(X) = \dfrac{k}{\left(1+x\right)^{k+1}}$
- When $\lambda = 0$, $c = k$, it reduces to one – parameter paralogistic distribution with pdf as $f(X) = \dfrac{\alpha^{2}x^{\alpha-1}}{\left(x^{\alpha}+1\right)^{\alpha+1}}$

7. Estimation of the Parameters of TBXII $\left(\text{c,k,}\lambda\right)$

7.1. Maximum likelihood estimation

Let X_1 , X_2 , …… , X_n be n independent identically distributed (iid) continuous random variables from the TBXII$\left(\text{c,k,}\lambda\right)$ distribution, then likelihood function may be written as:

$$L\left(X;c,k,\lambda\right) = c^{n}k^{n}\left(\prod_{i=1}^{n}x_{i}\right)^{c-1}\left[\prod_{i=1}^{n}\left(1+x_{i}^{c}\right)\right]^{-k-1}\prod_{i=1}^{n}\left(1-\lambda+2\lambda\left(1+x_{i}^{c}\right)^{-k}\right)$$

The corresponding log-likelihood function is given by

$$\log L\left(\text{X};c,k,\lambda\right) = n\log c + n\log k + \left(c-1\right)\sum_{i=1}^{n}\log x_{i} - \left(k+1\right)\sum_{i=1}^{n}\log\left(1+x_{i}^{c}\right) + \sum_{i=1}^{n}\log\left[1-\lambda+2\lambda\left(1+x_{i}^{c}\right)^{-k}\right] \qquad (16)$$

The derivation of the equation (12) for c, k and λ , and then

$$\frac{\partial \ell(c,k,\lambda)}{\partial c}=\frac{n}{c}+\sum_{i=1}^{n}\log x_i-(k+1)\sum_{i=1}^{n}\frac{x_i^c\log x_i}{\left(1+x_i^c\right)}$$

$$-2k\lambda\sum_{i=1}^{n}\frac{x_i^c\left(1+x_i^c\right)^{-(k+1)}\log x_i}{1-\lambda+2\lambda\left(1+x_i^c\right)^{-k}}=0 \qquad (17)$$

$$\frac{\partial \ell(c,k,\lambda)}{\partial k}=\frac{n}{k}+\sum_{i=1}^{n}\log\left(1+x_i^c\right)-2\lambda\sum_{i=1}^{n}\frac{\left(1+x_i^c\right)^{-k}\log\left(1+x_i^c\right)}{1-\lambda+2\lambda\left(1+x_i^c\right)^{-k}}=0 \qquad (18)$$

$$\frac{\partial \ell(c,k,\lambda)}{\partial \lambda}=\sum_{i=1}^{n}\frac{-1+2\left(1+x_i^c\right)^{-k}}{1-\lambda+2\lambda\left(1+x_i^c\right)^{-k}}=0 \qquad (19)$$

Because the above equations are non-linear system of equations, they can only be solved to estimate the three parameters numerically using an algorithm such as the Newton-Raphson iteration method .To illustrate this method we used R software to generate a sample of 300 observations from the TBXII distribution $\text{TBXII}(\text{c,k,}\lambda)$ with parameter values equal c=2, k=3, and λ=0.5. Figure (4) exhibits the histogram of randomly generated sample (n=300) from the $\text{TBXII}(\text{c,k,}\lambda)$.

Now, the three parameters of the $\text{TBXII}(\text{c,k,}\lambda)$ distribution have been estimated from the generated data set. Table(1) shows estimates of the parameters of a generated sample from TBXII with (n = 300 , c = 2 , k = 3 , λ=0.5) distribution using the MLE method.

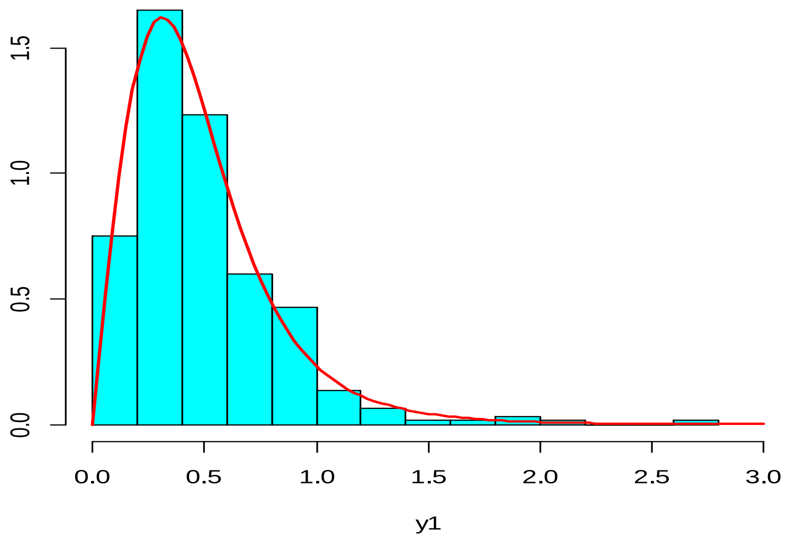

Figure (4): Histogram of generated sample from $\text{TBXII}(c,k,\lambda)$ distribution with the parameters $c=2$, $k=3$, $\lambda=0.5$

Table (1): MLE method of estimating parameters of a generated sample from TBXII(n = 300 , c = 2 , k = 3 , λ=0.5) distribution

Parameter	Estimate
c	1.9924509
k	2.3881318
λ	0.7909427
AIC: 74.82309	**BIC**: 85.93444

7.2. Estimation by the method of moments

Let X_1 , X_2 , …… , X_n be n independent identically distributed (iid) continuous random variables from the

$\text{TBXII}(\text{c,k},\lambda)$ distribution then the method of moments estimate is implies equating the empirical moments with the theoretical moments. These moments are defined by:

$$E\left(X^{i}\right)=m(j)\,, \qquad where \qquad m(j)=\frac{1}{n}\sum_{i=1}^{n}x_i^{j}\ , \qquad j=1,2,3$$

$$\therefore\quad E(X)=m(1)=\frac{1}{n}\sum_{i=1}^{n}x_i=\overline{\mathrm{X}}$$

Equating the empirical moments with the theoretical moments yields

$$(1-\lambda)\frac{\Gamma\left(k-\frac{1}{c}\right)\Gamma\left(\frac{1}{c}\right)}{c\,\Gamma(k)}+\lambda\frac{\Gamma\left(2k-\frac{1}{c}\right)\Gamma\left(\frac{1}{c}\right)}{c\,\Gamma(2k)}=\frac{1}{n}\sum_{i=1}^{n}x_i$$

$$\frac{(1-\lambda)\Gamma(2k)\Gamma\left(k-\frac{1}{c}\right)\Gamma\left(\frac{1}{c}\right)+\lambda\Gamma(k)\Gamma\left(2k-\frac{1}{c}\right)\Gamma\left(\frac{1}{c}\right)}{c\,\Gamma(k)\Gamma(2k)}=\frac{1}{n}\sum_{i=1}^{n}x_i$$

$$\frac{n(1-\lambda)\Gamma(2k)\Gamma\left(k-\frac{1}{c}\right)\Gamma\left(\frac{1}{c}\right)+n\lambda\Gamma(k)\Gamma\left(2k-\frac{1}{c}\right)\Gamma\left(\frac{1}{c}\right)}{c\sum_{i=1}^{n}x_i}=\Gamma(k)\Gamma(2k) \qquad (20)$$

and $\quad E\left(X^{2}\right)=m(2)=\frac{1}{n}\sum_{i=1}^{n}x_i^{2}$

Compensation for the value of $E(X^2)$ then

$$\frac{2(1-\lambda)\Gamma\left(k-\frac{2}{c}\right)\Gamma\left(\frac{2}{c}\right)}{c\,\Gamma(k)}+\frac{2\lambda\Gamma\left(2k-\frac{2}{c}\right)\Gamma\left(\frac{2}{c}\right)}{c\,\Gamma(2k)}=\frac{1}{n}\sum_{i=1}^{n}x_i^{2}$$

$$\frac{2(1-\lambda)\Gamma(2k)\Gamma\left(k-\frac{2}{c}\right)\Gamma\left(\frac{2}{c}\right)+2\lambda\Gamma(k)\Gamma\left(2k-\frac{2}{c}\right)\Gamma\left(\frac{2}{c}\right)}{c\,\Gamma(k)\Gamma(2k)}=\frac{1}{n}\sum_{i=1}^{n}x_i^{2}$$

$$\frac{2n(1-\lambda)\Gamma(2k)\Gamma\left(k-\frac{2}{c}\right)\Gamma\left(\frac{2}{c}\right)+2n\lambda\Gamma(k)\Gamma\left(2k-\frac{2}{c}\right)\Gamma\left(\frac{2}{c}\right)}{c\sum_{i=1}^{n}x_i^2}=\Gamma(k)\Gamma(2k) \qquad (21)$$

and $\quad E\left(X^3\right)=m(3)=\frac{1}{n}\sum_{i=1}^{n}x_i^3$

Compensation for the value of $E(X^3)$ then

$$\frac{3(1-\lambda)\Gamma\left(k-\frac{3}{c}\right)\Gamma\left(\frac{3}{c}\right)}{c\,\Gamma(k)}+\frac{3\lambda\Gamma\left(2k-\frac{3}{c}\right)\Gamma\left(\frac{3}{c}\right)}{c\,\Gamma(2k)}=\frac{1}{n}\sum_{i=1}^{n}x_i^3$$

$$\frac{3(1-\lambda)\Gamma(2k)\Gamma\left(k-\frac{3}{c}\right)\Gamma\left(\frac{3}{c}\right)+3\lambda\Gamma(k)\Gamma\left(2k-\frac{3}{c}\right)\Gamma\left(\frac{3}{c}\right)}{c\,\Gamma(k)\Gamma(2k)}=\frac{1}{n}\sum_{i=1}^{n}x_i^3$$

$$\frac{3n(1-\lambda)\Gamma(2k)\Gamma\left(k-\frac{3}{c}\right)\Gamma\left(\frac{3}{c}\right)+3n\lambda\Gamma(k)\Gamma\left(2k-\frac{3}{c}\right)\Gamma\left(\frac{3}{c}\right)}{c\sum_{i=1}^{n}x_i^3}=\Gamma(k)\Gamma(2k) \qquad (22)$$

Because equations (20) , (21) and (22) are non-linear system of equations, they can only be solved to estimate the three parameters c , k and λ numerically using an algorithm such as the Newton-Raphson iteration method . To illustrate this method we used the above generated sample of 300 observations from the TBXII distribution $\text{TBXII}(c,k,\lambda)$ with parameter values equal c=2, k=3, and λ=0.5. The three parameters of the $\text{TBXII}(c,k,\lambda)$ distribution have been estimated from the generated data set using the moments matching method. Table(2) shows estimates of the parameters of the generated sample from TBXII with (n = 300 , c = 2 , k = 3 , λ=0.5) distribution using the moments matching method.

Table (2): Moment matching estimates method of estimating parameters of a generated sample from TBXII (n = 300 , c = 2 , k = 3 , λ=0.5) distribution.

Parameter	Estimate

c	1.9886866
k	3.0197227
λ	0.4601886
AIC: 75.40785 **BIC**: 86.51919	

7.3. Estimation by Quantile Matching Method

A quantile for a random variable X is a constant, x_q , such that $P\left(X \le x_q\right) = q$. The quantile x_q of the $TBXII\left(c,k,\lambda\right)$ distribution is defined as follows:

$$q = F(X) = \left[1+\lambda\left(1+x_q^c\right)^{-k}\right]\left[1-\left(1+x_q^c\right)^{-k}\right]$$

$$q = 1-\left(1+x_q^c\right)^{-k} + \lambda\left(1+x_q^c\right)^{-k} - \lambda\left(1+x_q^c\right)^{-2k}$$

$$\Rightarrow \left(q-1\right)+\left(1-\lambda\right)\left(1+x_q^c\right)^{-k} + \lambda\left(1+x_q^c\right)^{-2k} = 0$$

Assuming that $y = (1 + x^c)^{-k}$, then $\left(q-1\right)+\left(1-\lambda\right)y+\lambda y^2 = 0$ and solving for y we have :

$$y = \frac{\left(\lambda-1\right) \pm \sqrt{\left(1-\lambda\right)^2 - 4\lambda\left(q-1\right)}}{2\lambda}$$

Now ,

$$\frac{\left(\lambda-1\right) \pm \sqrt{\left(1-\lambda\right)^2 - 4\lambda\left(q-1\right)}}{2\lambda} = \left(1+x_q^c\right)^{-k}$$

$$\left(\frac{\left(\lambda-1\right) \pm \sqrt{\left(1-\lambda\right)^2 - 4\lambda\left(q-1\right)}}{2\lambda}\right)^{\frac{-1}{k}} = 1+x_q^c$$

$$\left(\frac{\left(\lambda-1\right) \pm \sqrt{\left(1-\lambda\right)^2 - 4\lambda\left(q-1\right)}}{2\lambda}\right)^{\frac{-1}{k}} - 1 = x_q^c$$

$$\therefore\; x_q = \left[\left(\frac{(\lambda-1) \pm \sqrt{(1-\lambda)^2 - 4\lambda(q-1)}}{2\lambda} \right)^{\frac{-1}{k}} - 1 \right]^{\frac{1}{c}} \tag{23}$$

When q = 0.5 , We obtain the median that was calculated in the equation (12).

Now, denote the 100q-th quantile of the distribution by $X_q(\theta)$ for which in the case of continuous distribution is the solution to: $F(X_q(\theta)|\theta) = q$. Denote the smoothed empirical estimate by x_q. Here $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ denote the order statistics of the sample. The quantile matching estimate of θ is any solution to the *p* equations:

$$X_{qk}(\hat{\theta}) = x_{qk}, \quad for\ k = 1, 2, \ldots, p \tag{24}$$

where the qk's are p arbitrarily chosen quantiles. For further details see Sgouropoulos,Yao& Yastremiz (2015).

Since equation (24) for different values of k produces a non-linear system of equations, they can be solved to estimate the three parameters *c* , *k* and λ numerically using an algorithm such as the Newton-Raphson iteration method. To illustrate this method we used the above generated sample of 300 observations from the TBXII distribution $\mathrm{TBXII}(c,k,\lambda)$ with parameter values equal $c=2, k=3, \text{ and } \lambda=0.5$. The three parameters of the $\mathrm{TBXII}(c,k,\lambda)$ distribution have been estimated from the generated data set using the moments matching method. Table(3) shows estimates of the parameters of the generated sample from TBXII with (n = 300 , c = 2 , k = 3 , $\lambda=0.5$) distribution using the quantile matching estimation method.

Table (3): Quantile matching estimates method of estimating parameters of a generated sample from TBXII(n = 300 , c = 2 , k = 3 , λ=0.5) distribution

Parameter	Estimate
c	2.0121828
k	2.1627008
λ	0.9998793
AIC: 80.96509 **BIC**: 92.07643	

7.4. Estimation by the maximum goodness-of-fit Method

The maximum goodness-of-fit estimation(MGF) method was first studied by Wolfowitz (1953 &1957) and Kac, et al. (1955). Subsequently, Pollard (1980) proved the $\sqrt{n}$ -consistency of the mgfs and found its asymptotic distribution. We assume that we have a complete sample ($X_1, X_2, \ldots, X_n$) of n iid observations on a TBXII having unknown parameters c, k, and λ. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the corresponding order statistics and S_n(x) be the empirical distribution function (EDF) (see Rao, 1973). The'distance' between F(x) and S_n(x) can be measured using EDF statistics such as the Kolmogorov–Smirnov's (KS) statistic

$$D_n = \sup_x |F(x) - S_n(x)| \tag{25}$$

or the statistics resulting from

$$n\int_{-\infty}^{\infty} \psi(x)\left[F(x) - S_n(x)\right]^2 dF(x)$$

for different weight functions ψ(x) ≥ 0. The KS, Crame–von Mises (CM), Anderson–Darling (AD), and AD of second degree (AD2) statistics assign the same weight to both tails of the distribution, but this weight is larger for the AD2 statistic and progressively smaller for the AD and CM statistics. Moreover, by using the right- or left-tail AD statistics, one can assign more weight to one tail than to the other tail. For further details see for example Luceno (2008). MGFs for the parameters c, k and λ can be obtained by numerically minimizing any one of the empirical distribution function(EDF) statistics provided in Eq. (6) and (7).

The maximization process of (25) can be carried out to estimate the three parameters c , k and λ numerically using an algorithm such as the Newton-Raphson iteration method. To illustrate this method we used the above generated sample of 300 observations from the TBXII distribution $\text{TBXII}(c,k,\lambda)$ with parameter values equal $c=2$, $k=3$, and $\lambda=0.5$. The three parameters of the $\text{TBXII}(c,k,\lambda)$ distribution have been estimated from the generated data set using the goodness-of-fit estimation(MGF) method. Table(4) shows the estimates of the parameters of the generated sample from TBXII with ($n = 300$, $c = 2$, $k = 3$, $\lambda=0.5$) distribution using the MGF method.

Table (4): Maximum goodness-of-fit estimates method of estimating parameters of a generated sample from TBXII($n = 300$, $c = 2$, $k = 3$, $\lambda=0.5$) distribution

Parameter	**Estimate**
C	2.0298768
K	3.0197235
λ	0.5004902
AIC: 75.29880 **BIC**: 86.41015	

8. Comparison between different estimates

To compare between the four different estimation methods we used the above generated sample of 300 observations from the TBXII distribution (c, k, λ) with parameter values equal $c=2$, $k=3$, $\lambda=0.5$. The goodness of fit criteria have been calculated from the generated data set using the four different estimation methods . Table(5) shows the goodness of fit criteria of the generated sample from TBXII (n=300, c=2 , k=3 , λ=0.5) distribution using the four different estimating methods . It is clear that the values of these criteria are very close for the four estimation methods .

Table (5): Goodness of fit tests for generated data using the four methods of estimating parameters .

Method	mle	mme	qme	mge
Kolmogorov-Smirnov statistic	0.04173086	0.03776357	0.04456320	0.04456320
Cramer-von Mises statistic	0.04625034	0.04760333	0.08197766	0.04283697
Anderson-Darling statistic	0.28340472	0.31469843	0.71476508	0.27916521
Akaike's Information Criterion	74.82309	75.40785	80.96509	75.29880
Bayesian Information Criterion	85.93444	86.51919	92.07643	86.41015

Figures (5) , (6) ,(7) and (8) also illustrate the goodness of fit for the estimated distribution using the four estimation methods of the parameters to the generated data set. Plots of the empirical and theoretical CDFs, histogram and theoretical densities,P-P plot and Q-Q plot indicate that the four fitted distributions provide a very good and close fit to the dataset .

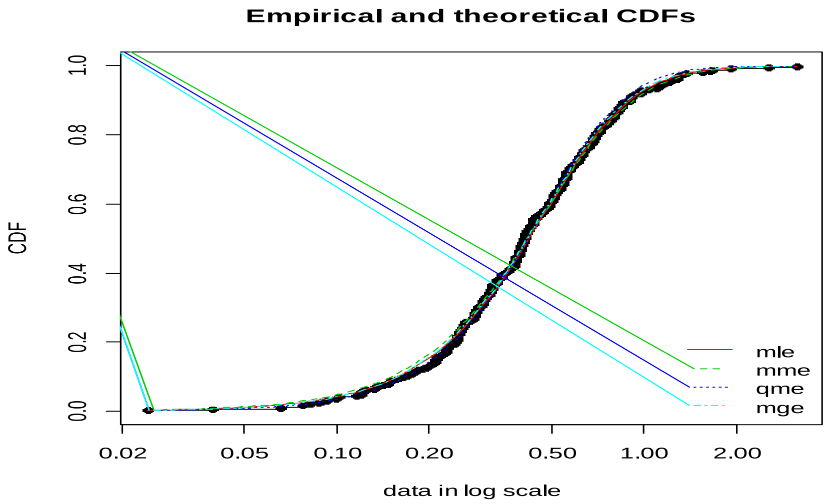

Figure (5): Empirical and theoretical CDFs using the four different methods of estimating .

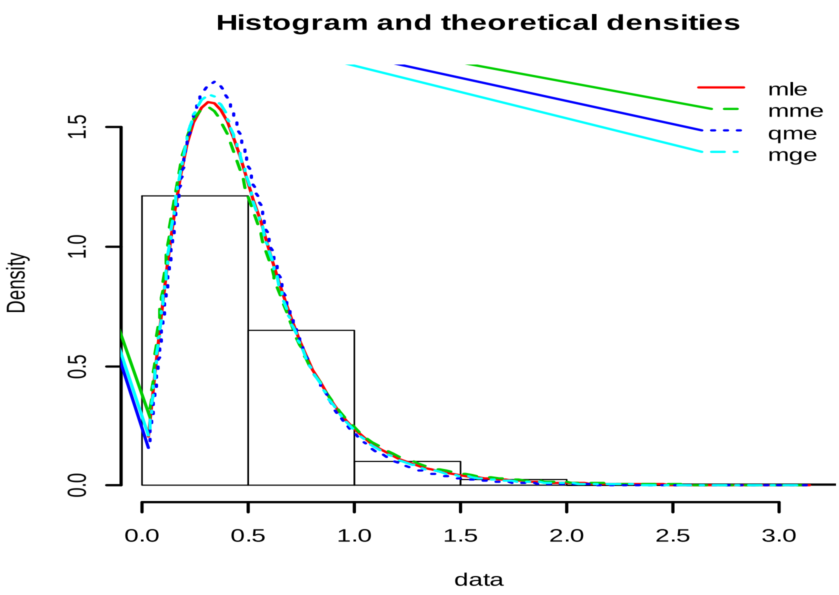

Figure (6): Histogram and theoretical densities using the four different methods of estimating

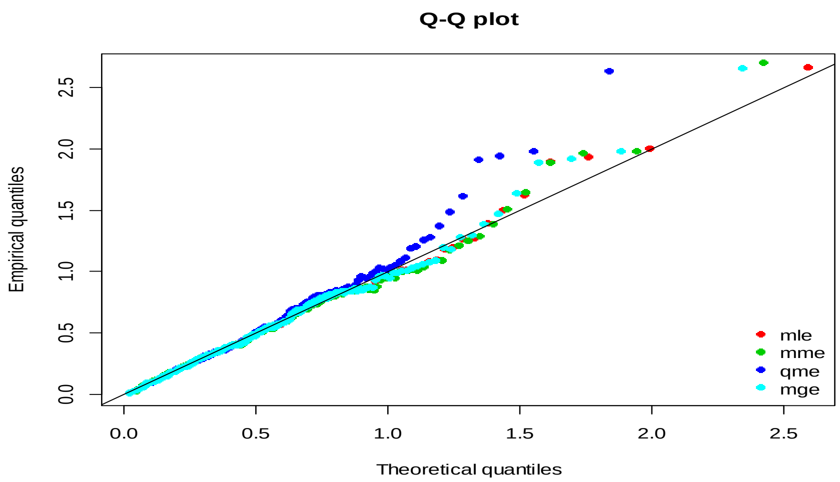

Figure (7): Q-Q plot for theoretical and empirical quantiles using the four different methods of estimation .

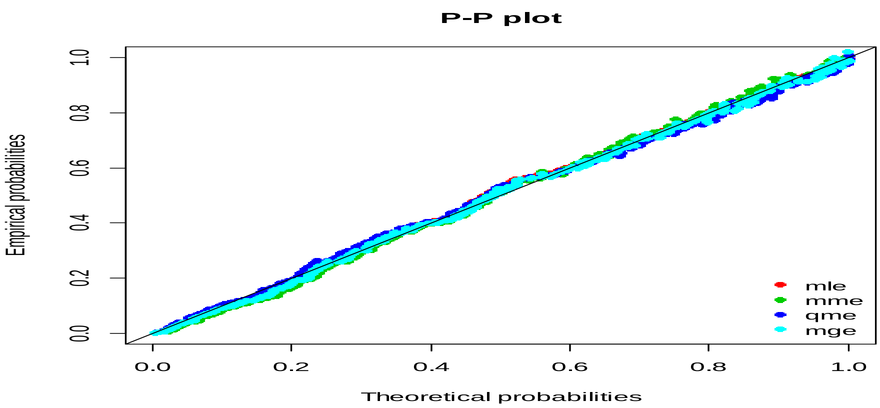

Figure (8): P-P plot for theoretical and empirical quantiles using the four different methods of estimation.

9. Distributional Properties of the estimates of the parameters - a simulated study

We used R software to simulate samples of different sizes from the TBXII distribution TBXII (c, k, λ) with many cases of parameter values , then to estimate the standard errors of the estimates of the three parameters using the four different estimation methods . Table (6) shows results of simulation study to estimate the standard errors of the estimates of the three parameters values equal c=2 , k=3 and λ=0.5. Table (7) shows results of simulation study to estimate the standard errors of the estimates of the three parameters values equal c=3 , k=1 and λ=1. And table (8) shows results of simulation study to estimate the standard errors of the estimates of the three parameters values equal c=5 , k=2 and λ=1. As can be seen in the three tables above , the standard error decreases as the sample size increase for the four methods of estimation (i.e. the estimators of the parameters c , k and λ are consistent) . This means that the MLE estimators , the MME estimators , the QME estimators , and the MGE estimators are all consistent estimators .

10. Application of the Transmuted Burr Type XII distribution to Lifetime of Breast Cancer Patients' Data in Gaza Strip

The Burr XII distribution has been shown to be very important in modeling heavy tailed data since it offers greater flexibility than many other lifetime distributions. In this section, we apply the TBXII distribution to the survival time of breast cancer patients in Gaza Strip. We believe that these data are heavy tailed and can be a good example in which the TBXII distribution can be useful because patients of this fatal disease in this poor area receive different types of treatment locally and abroad. The data have been obtained from the Ministry of Health in Gaza City on the incidence dates and death dates of about 1,000 breast cancer patients within a period of 5 years starting from beginning of 2009 to end of 2013. The survival times for those patients were computed. Among them, 703 people were still alive at the end of 2013 and 55 patients had a zero lifetime and were believed to be wrongly reported or their records were absent upon death and thus excluded from the analysis. The remaining 242 patients have a lifetime as shown in Table (9). Fig. (9) exhibits a histogram of the above data where it is clear that the distribution of the data is so close to the *transmuted Burr type XII* distribution .

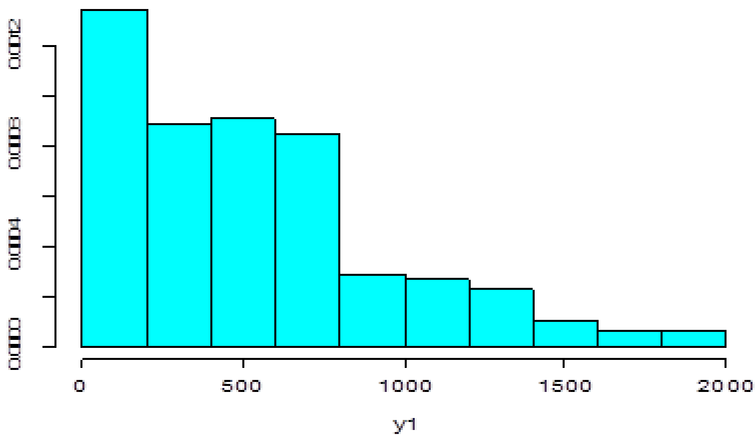

Figure (9) : The histogram of the breast cancer patient's data

10.1 Estimation of the parameters of the Transmuted Burr Type XII(TBXII) distribution from the lifetime breast cancer patients' data:

Since the $\text{TBXII}(c,k,\lambda)$ distribution takes small values as it appears in the discussion of the previous chapters and in Fig. (4.2) and in order to be able to estimate the parameters of this distribution the data has been divided by 700 in order to have a maximum value of around 3.0. After that, if the breast cancer patients in Gaza Strip data follow the $\text{TBXII}(c,k,\lambda)$ distribution then the estimates of three parameters using the R software are :

c= 1.5454243 , k= 0.8699955 , λ = 1.0000000

10.2 Expected lifetime of breast cancer patients in Gaza Strip:

Expected mean survival time of breast cancer patients based on these data and the $\text{TBXII}(c,k,\lambda)$ distribution using the R software is 534.9421 day which is roughly a year and a half; and the variance and standard deviation are 180206.2 and 424.507 respectively . The mean survival time of breast cancer patients seem to be high due to several reasons, including the type of cancer and benign or malignant nature and type of treatment.

Figure (10) compares the estimated kernel density estimate of the distribution of the breast cancer patient's data (divided by 700) with a (black) solid line and the theoretical distribution of $TBXII(c,k,\lambda)$ with a (red) solid line and that shows how close the distribution of the original data to the theoretical $TBXII(c,k,\lambda)$ distribution. but the figure shows that we need to add a fourth parameter for distribution to be more suitable for data .

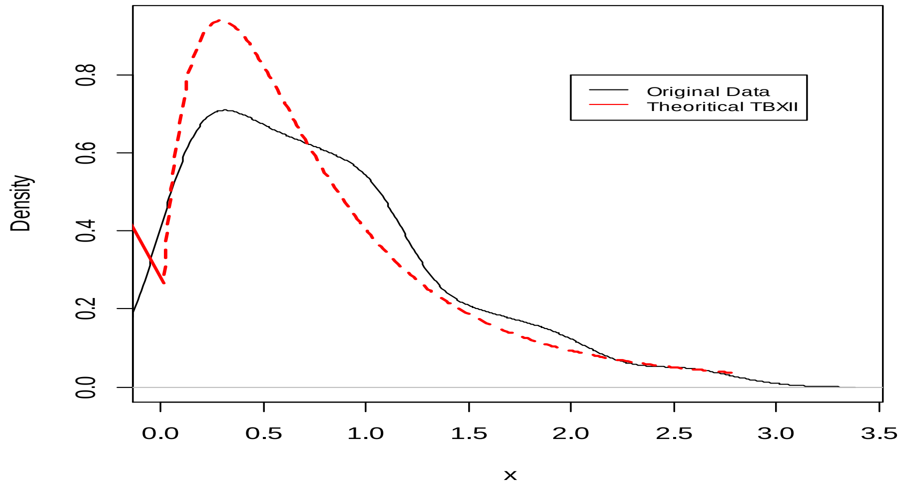

Figure (10): Kernel density estimate of breast cancer patient's data (black line) and the theoretical distribution of $TBXII(c,k,\lambda)$ (red line).

11. Conclusion

Based on all the above illustrations of the importance of the transmuted Burr XII distribution in applied statistics, especially in lifetime data analysis, we may conclude the following points :

1. The transmuted Burr XII distribution provides a very good fit to breast cancer patients' data in Gaza Strip.

2. The analysis of breast cancer patients' data supported all the theatrical results.

3. The expected mean survival time of breast cancer patients based on the data and the transmuted Burr XII distribution is 534.9421 days.

4. The transmuted Burr XII is important to use in many applications .

5. The transmuted Burr XII distribution should be used to fit survival and lifetime data in survival analysis and many applications such as economics, finance , quality assurance, health and environmental applications.

Table (6): Results of a simulation study to estimate the standard errors of the estimates of the three parameters (c= 2 , k=3 and λ = 0.5).

Method	n	Mean			Standard Errors		
		c	k	λ	Sd(c)	Sd(k)	Sd(λ)
mle	50	1.996776	2.528549	0.7869137	0.217572	0.6485235	0.2739355
	100	1.96736	2.475906	0.779251	0.1500494	0.5139198	0.2488957
	150	1.961836	2.454925	0.7827137	0.1279376	0.5062599	0.2256309
	200	1.96281	2.457965	0.7717604	0.1116613	0.4854807	0.2303225
	300	1.956122	2.418412	0.7835168	0.09476395	0.4615201	0.2170381
mme	50	2.016145	3.357181	0.3530912	0.2244868	0.7157382	0.4331666
	100	1.95555	3.501207	0.2484325	0.1658196	0.6815333	0.3437598
	150	1.93879	3.483149	0.2377761	0.1348461	0.6242901	0.31329
	200	1.938043	3.48099	0.2310547	0.118858	0.6020001	0.2992596
	300	1.928708	3.44609	0.2381491	0.1025184	0.5932596	0.2894274
qme	50	2.063992	2.93921	0.7118178	0.3405154	1.212981	0.2614968

	100	2.025728	2.780276	0.7205913	0.2399305	0.8118119	0.2563672
	150	2.005912	2.72451	0.7087405	0.1885185	0.673146	0.255157
	200	1.993816	2.664259	0.7164086	0.1645357	0.6406213	0.2550557
	300	1.983865	2.625469	0.7125982	0.1291349	0.5472609	0.2541149
mge	50	2.009724	2.705076	0.7343732	0.2615949	0.8237504	0.2561202
	100	1.985631	2.626377	0.7397021	0.1860084	0.6598685	0.2296795
	150	1.978528	2.608076	0.7318303	0.1531048	0.5967107	0.2229366
	200	1.975147	2.568354	0.7373803	0.1288624	0.5451762	0.220152
	300	1.971389	2.566385	0.7267205	0.109838	0.5137272	0.219509

Table (7): Results of a simulation study to estimate the standard errors of the estimates of the three parameters (c= 3 , k= 1 and $\lambda = 1$)

Method	n	Mean			Standard Errors		
		c	k	λ	Sd(c)	Sd(k)	Sd(λ)
mle	50	3.156518	1.871018	0.1967492	0.3756468	0.3733899	0.2699135
	100	3.105262	1.838061	0.2001578	0.260083	0.3055403	0.260066
	150	3.091353	1.821965	0.2029004	0.2184359	0.2893193	0.2575209
	200	3.087968	1.823306	0.1918368	0.1885195	0.2785901	0.2508146
	300	3.067441	1.853094	0.1531433	0.1563927	0.2390218	0.2236538
mme	50	3.314313	2.007429	0.07455539	0.4018516	0.3707933	0.2095028
	100	3.191125	1.974525	0.07076558	0.2874448	0.2591879	0.1816491
	150	3.154316	1.963386	0.0666959	0.234071	0.2261605	0.167068
	200	3.149733	1.952608	0.06651544	0.2003894	0.2084292	0.1646924
	300	3.133506	1.93989	0.07135368	0.171816	0.2012995	0.162779
qme	50	3.2027	1.5545	0.4947814	0.527528	0.4521055	0.3471338

	100	3.140299	1.441598	0.5797239	0.3744475	0.3723014	0.3474804
	150	3.110218	1.404575	0.5952009	0.2950125	0.3350325	0.3367356
	200	3.090663	1.356702	0.6336066	0.2576116	0.3347949	0.3332156
	300	3.076442	1.314125	0.6636529	0.2013189	0.3035459	0.3132903
mge	50	3.146289	1.773246	0.2538592	0.4246781	0.352574	0.2815457
	100	3.100502	1.761452	0.2528023	0.3012548	0.3180796	0.2797301
	150	3.086705	1.75822	0.2450499	0.2464688	0.3003041	0.2738041
	200	3.080106	1.777488	0.2160764	0.2076169	0.2739097	0.2546242
	300	3.068367	1.793429	0.1932491	0.1738254	0.2534521	0.2398721

Table (8): Results of a simulation study to estimate the standard errors of the estimates of the three parameters (c=5 , k= 2 and λ = 1)

Method	n	Mean			Standard Errors		
		c	k	λ	Sd(c)	Sd(k)	Sd(λ)
mle	50	5.261577	3.097257	0.5824939	0.5874732	1.09891	0.4022748
	100	5.17801	3.114395	0.5247221	0.4058861	0.9453316	0.3955676
	150	5.159479	3.217889	0.4567324	0.3471089	0.8862652	0.3942523
	200	5.152713	3.220356	0.4422662	0.2996251	0.8666692	0.3908044
	300	5.121602	3.284551	0.394748	0.2505476	0.8452979	0.3949872
mme	50	5.310082	2.423931	0.8838666	0.5706511	0.4381035	0.1358351
	100	5.215118	2.325294	0.9000077	0.3913554	0.3103047	0.0750477
	150	5.199995	2.311937	0.8958921	0.3236715	0.2646101	0.06743156
	200	5.201152	2.294661	0.8982521	0.2861996	0.2347842	0.06043358
	300	5.18399	2.274676	0.8999935	0.2420487	0.2017757	0.06036843

qme	50	5.255875	2.866248	0.6805296	0.717484	1.043374	0.2901613
	100	5.212757	2.737786	0.7170803	0.5843326	0.8839497	0.2771616
	150	5.177578	2.635419	0.7324588	0.478326	0.685604	0.2543637
	200	5.150488	2.58514	0.7374734	0.4278889	0.6520412	0.2535985
	300	5.126557	2.528722	0.7438356	0.3400084	0.5514511	0.2463091
mge	50	5.23989	3.38346	0.3943276	0.6869841	0.8533886	0.363735
	100	5.163379	3.265005	0.421923	0.4868004	0.8026716	0.3630988
	150	5.142723	3.258676	0.4098215	0.4011806	0.7822901	0.3604771
	200	5.131753	3.218655	0.417606	0.3373853	0.7819923	0.3696023
	300	5.112675	3.168168	0.4287168	0.2839363	0.7751464	0.3698251

Table (9) : Data of lifetime (days) of breast cancer patients in Gaza Strip

11	73	173	283	402	516	674	779	1142
17	82	175	284	403	520	676	784	1147
25	89	177	290	403	530	679	786	1156
27	89	177	292	406	532	680	795	1164
27	89	178	293	406	537	682	822	1212
29	90	178	294	422	538	686	830	1231
29	104	181	298	423	543	698	831	1238
29	106	189	300	434	548	699	837	1267
29	118	190	302	439	555	716	848	1285
29	118	196	314	444	558	716	889	1307
30	118	198	323	449	562	722	898	1332

31	118	205	323	457	574	722	906	1361
34	139	207	325	457	578	726	912	1364
38	145	207	326	463	583	726	916	1368
38	145	213	334	466	591	729	933	1380
41	145	219	337	468	593	732	962	1412
47	146	227	339	470	597	735	992	1416
49	147	231	347	482	607	742	995	1456
57	148	234	347	484	609	745	1004	1554
57	148	234	351	487	616	749	1029	1597
59	148	236	354	494	616	752	1048	1614
59	158	236	355	498	629	752	1052	1756
59	160	237	362	499	646	764	1053	1790
59	161	251	363	499	647	764	1069	1813
60	161	257	367	502	653	770	1091	1835
65	164	272	367	503	663	771	1120	1997
66	168	280	379	504	668	776	1140	

References

[1] Arnold. B.C., Balakrishnan. N. and Nagaraja, H.N. (1992), A first course in order statistics, Willy, New York .
[2] Burr , I .W. (1942), Cumulative frequency functions , Annals of Mathematical Statistics , 13, 215-232 .
[3] Luceno , A. (2008) Maximum likelihood vs. maximum goodness of fit estimation of the three-parameter Weibull distribution, Journal of Statistical Computation and Simulation, 78:10, 941-949.
[4] Kac, M., Kiefer, J. and Wolfowitz, J. (1955), On tests of normality and other tests of goodness of fit based on distance methods. The Annals of Mathematical Statistics, 26, 189–211.
[5] Pollard, D. (1980), The minimum distance method of testing. Metrika, 27, 43–70.
[6] Rao, C.R. (1973), Linear Statistical Inference and Its Applications 2nd ed. NewYork: Wiley.
[7] Sgouropoulos, N., Yao, Q. &Yastremiz, C. (2015) Matching a Distribution by Matching Quantiles Estimation, Journal of the American Statistical Association,110:510, 742-759.
[8] Shaw, W. and Buckley, I. (2007), The Alchemy of Probability Distributions : Beyond Gram- Chalier expansions , and a skew – kurtotic-normal distribution from a rank transmutation map . Research report.
[9] Wolfowitz, J. (1953), Estimation by the minimum distance method. Annals of the Institute of Statistical Mathematics, 5, 9–23.
[10] Wolfowitz, J. (1957), The minimum distance methods. The Annals of Mathematical Statistics, 28, 75–88.