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Generating Function Relations for Modified Hypergeometric Functions by Lie-theoretic Method

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Abstract:

In this paper we derived several generating relations involving the modified hypergeometric functions ${}_2F_1(a,b;c+n;x)$ *by the group theoretical method known as Wisner's method. We have considered a three parameter Lie group by giving a suitable interpretation to the index* n *of the modified hypergeometric functionsand obtained some known as well as some new generating relation for modified hypergeomtric functions. Some particular cases of these relations are also investigated.*

Keywords: Hypergeometric functions, Generating relations, Lie algebra

Mathematics subject classification: 33C05, 33C45, 33C80

1 Introduction

Several generating relations for hypergeometric functionshave been derived by different methods e.g. classical, theory of Lie-group etc. see [3,6]. In this paper we appeal to the Lie theoretic concepts and derive generating functions involving series of the modified Hypergeometric functions ${}_2F_1(a,b;c+n;x)$, because of the important role which hypergeometric functionsplay in problems of physics and applied mathematics, the theory of generating functions has been developed various directions and found wide applications in different branches of science and technology. Weisner in [7] introduce Lie operators and then obtained generating functions for generalized hypergeometric corresponding to increasing and decreasing of numerator parameter a. Monoch and Jain [4], Agrawal and Jain [1] etc, have also applied it to obtain generating function by variation of parameter a. Also Chongdar [2] has derived some generating functions for the said functionsby Lie algebraic method. This paper is an attempt to exhibit the group theoretic significance of generating relations for modified hypergeometric functions.

We begin by considering the following ordinary differential equation which satisfed by the Hypergeometric functions$y={}_2F_1(a,b;c+n;x)$ [5].

$$x(1-x)\frac{d^2y}{dx^2}+[c+n-(a+b+1)x]\frac{dy}{dx}-aby=0 \quad (1.1)$$

2 Group-Theoretic Discussion

Replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, n by $y\frac{\partial}{\partial y}$ and y by $u(x,y)$ in (1.1), we get the following partial differential equation :

$$x(1-x)\frac{\partial^2 u}{\partial x^2}+[c-(a+b+1)x]\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\,\partial y}-abu=0 \quad (2.1)$$

Thus $u(x,y)=y^n{}_2F_1(a,b;c+n;x)$ is a solution of (2.1), since ${}_2F_1(a,b;c+n;x)$ is a solution of (1.1).

Now we introduce the following linearly differeintal operaters

$$\begin{cases} J^3=y\frac{\partial}{\partial y}+\frac{1}{2}(2c-a-b-1) \\ J^-=y^{-1}[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+c-1] \\ J^+=y[(1-x)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-(a+b-c)] \end{cases} \quad (2.2)$$

such that

$$J^3(y^n{}_2F_1(a,b;c+n;x))=(n+\frac{1}{2}(2c-a-b-1))y^n{}_2F_1(a,b;c+n;x)$$

$$J^-(y^n{}_2F_1(a,b;c+n;x))=(c+n-1)y^{n-1}{}_2F_1(a,b;c+n-1;x)$$

$$J^{+}(y^{n}{}_{2}F_{1}(a,b;c+n;x))=\frac{(c+n-a)(c+n-b)}{(c+n)}y^{n+1}{}_{2}F_{1}(a,b;c+n+1;x)$$

Now we have the following commutator relations:

$$[J^3,J^{\pm}]=\pm J^{\pm} \quad ; \qquad [J^{+},J^{-}]=2J^3 \qquad (2.3)$$

where $[A,B]u=(AB-BA)u$.

The above commutator relations (2.3) show the set of operators $J^3, J^{-}, J^{+}, 1$ where 1 stands for identity operator, generates a Lie algebra which isomorphic to $sl(2)$.

Now we can write the differential operator as follow

$$L\equiv J^{-}J^{+}-(J^3-\tfrac{1}{2}(a-b-1))(J^3+\tfrac{1}{2}(a-b+1)) \quad (2.4)$$

from which it follows that L commutes with each operator

$$[J^3,L]=[J^{-},L]=[J^{+},L]=0 \qquad (2.5)$$

3 Extended forms of the groups generated by J-operators

To find the extended form of the group generated by J^3, we must solve the following differentail equations

$$\frac{\partial y(a\prime)}{\partial a\prime}=y(a') \qquad (3.1)$$

and

$$\frac{\partial v(a\prime)}{\partial a\prime}=\frac{1}{2}(2c-a-b-1)v(a') \qquad (3.2)$$

where $y(0)=y$, and $v(0)=1$.

From (3.1) we get

$$\int\frac{\partial y(a\prime)}{\partial a\prime}\frac{1}{y(a\prime)}da'=\int da'$$

$$\log y(a')=a'+k$$

we put $a'=0$ to find the constant k

$$\log y=k$$

Then

$$\log y(a')=a'+\log y$$

$$y(a')=ye^{a'}$$

and from (3.2) we get

$$\int\frac{\partial v(a\prime)}{\partial a\prime}\frac{1}{v(a\prime)}da'=\int\frac{1}{2}(2c-a-b-1)da'$$

$$\log v(a') = \frac{1}{2}(2c-a-b-1)a' + k$$

we put $a' = 0$ to find the constant k

$$k = 0$$

Then

$$\log v(a') = \frac{1}{2}(2c-a-b-1)a'$$

$$v(a') = e^{\frac{1}{2}(2c-a-b-1)a'}$$

Thus

$$e^{a'J^3}u(x,y) = e^{\frac{1}{2}(2c-a-b-1)a'}u(x,ye^{a'}) \qquad (3.3)$$

Similarly we get

$$e^{b'J^-}u(x,y) = (1+\frac{b'}{y})^{c-1}u(x+\frac{xb'}{y},y+b') \qquad (3.4)$$

and

$$e^{c'J^+}u(x,y) = (1-c'y)^{(a+b-c)}u((x-1)(1-c'y)+1,\frac{y}{1-c'y}) \qquad (3.5)$$

Therefore

$$e^{c'J^+}e^{b'J^-}e^{a'J^3}u(x,y) = e^{\frac{1}{2}(2c-a-b-1)a'}(1-c'y)^{a+b-c}(1+\frac{b'}{y})^{c-1} \qquad (3.6)$$

$$\times u((x(1+\frac{b'}{y})-1)(1-c'(y+b'))+1,\frac{(y+b')e^{a'}}{1-c'(y+b')})$$

4 Generating Functions

We will obtain generating relations from the operator J^3 by considering the following two cases of the transformed functionexp$(c'J^+)$exp(b'^{J^-}) ${}_2F_1(a,b;c+n;x)$

From (2.1) $u(x,y) = y^n{}_2F_1(a,b;c+n;x)$ is a solution of the system

$$\begin{cases} Lu = 0 \\ (A - (n + \frac{1}{2}(2c - a - b - 1)))u = 0 \end{cases} \tag{4.1}$$

Since L commutes with the operators, we have

$$S(L)(y^n {}_2F_1(a, b; c + n; x)) = (L)S(y^n {}_2F_1(a, b; c + n; x)) = 0$$

where

$$S = e^{a'J^3} e^{c'J^+} e^{b'J^-}$$

Therefore the transformation $S(y^n {}_2F_1(a, b; c + n; x))$ is also annulled by xL

Now put $a' = 0$ in (3.6), we get

$$e^{c'J^+} e^{b'J^-} u(x, y) = (1 - c'y)^{a+b-c} (1 + \frac{b'}{y})^{c-1} \tag{4.2}$$

$$\times u((x(1 + \frac{b'}{y}) - 1)(1 - c'(y + b')) + 1, \frac{y + b'}{1 - c'(y + b')})$$

So, we obtian

$$e^{c'J^+} e^{b'J^-} [y^n {}_2F_1(a, b; c + n; x)] = (1 - c'y)^{a+b-c} (1 + \frac{b'}{y})^{c-1} (\frac{y + b'}{1 - c'(y + b')})^n \tag{4.3}$$

$$\times {}_2F_1(a, b; c + n; (x(1 + \frac{b'}{y}) - 1)(1 - c'(y + b')) + 1)$$

So we get the following special cases :-

Case I : Let $b' = 1$, $c' = 0$, (4.3) reduces to:

$$e^{J^-} [y^n {}_2F_1(a, b; c + n; x)] = (1 + \frac{1}{y})^{c-1} (y + 1)^n {}_2F_1(a, b; c + n; x(1 + \frac{1}{y})) \tag{4.4}$$

Now expanding this function, we get

$$e^{J^-} [y^n {}_2F_1(a, b; c + n; x)] = \sum_{m=0}^{\infty} \frac{(J^-)^m}{m!} [y^n {}_2F_1(a, b; c + n; x)]$$

$$= \sum_{m=0}^{\infty} \frac{(J^-)^{m-1}}{m!}\left[(c+n-1)y^{n-1}{}_2F_1(a,b;c+n-1;x)\right]$$

$$= \sum_{m=0}^{\infty} \frac{(J^-)^{m-m}(c+n-m)_m}{m!}\left[y^{n-m}{}_2F_1(a,b;c+n-m;x)\right]$$

Thus

$$e^{J^-}\left[y^n{}_2F_1(a,b;c+n;x)\right] = \sum_{m=0}^{\infty} \frac{(c+n-m)_m}{m!} y^{n-m}{}_2F_1(a,b;c+n-m;x) \tag{4.5}$$

Equating the two equations (4.4) and (4.5), we get

$$(1+\frac{1}{y})^{c+n-1}{}_2F_1(a,b;c+n;x(1+\frac{1}{y})) \tag{4.6}$$

$$= \sum_{m=0}^{\infty} \frac{(c+n-m)_m}{m!}\left[y^{-m}{}_2F_1(a,b;c+n-m;x)\right]$$

Let $t = \frac{1}{y}$

$$(1+t)^{c+n-1}{}_2F_1(a,b;c+n;x(1+t)) \tag{4.7}$$

$$= \sum_{m=0}^{\infty} \frac{(c+n-m)_m}{m!} \quad {}_2F_1(a,b;c+n-m;x)t^m$$

Case II : Let $b' = 0$, $c' = 1$, (4.3) reduces to:

$$e^{J^+}\left[y^n{}_2F_1(a,b;c+n;x)\right] = (\frac{y}{1-y})^n(1-y)^{a+b-c} \tag{4.8}$$

$$\times_2 F_1(a,b;c+n;(x-1)(1-y)+1)$$

Now expanding this function, we get

$$e^{J^+}\left[y^n{}_2F_1(a,b;c+n;x)\right] = \sum_{k=0}^{\infty} \frac{(J^+)^k}{k!}\left[y^n{}_2F_1(a,b;c+n;x)\right]$$

$$= \sum_{k=0}^{\infty} \frac{(J^+)^{k-1}}{k!} [\frac{(c+n-a)(c+n-b)}{c+n} y^{n+1} {}_2F_1(a,b;c+n+1;x)]$$

$$= \sum_{k=0}^{\infty} \frac{(J^+)^{k-k}(c+n-a)_k(c+n-b)_k}{(c+n)_k k!} [y^{n+k} {}_2F_1(a,b;c+n+k;x)]$$

Thus

$$e^{J^+}[y^n {}_2F_1(a,b;c+n;x)] \tag{4.9}$$

$$= \sum_{k=0}^{\infty} \frac{(c+n-a)_k(c+n-b)_k}{(c+n)_k k!} [y^{n+k} {}_2F_1(a,b;c+n+k;x)]$$

Equating the two equations (4.8) and (4.9) , we get

$$(1-y)^{a+b-c-n} {}_2F_1(a,b;c+n;(x-1)(1-y)+1) \tag{4.10}$$

$$= \sum_{k=0}^{\infty} \frac{(c+n-a)_k(c+n-b)_k}{(c+n)_k k!} \ {}_2F_1(a,b;c+n+k;x) y^k$$

Finally, we want to point out that the interchanging of the order of the operators in (3.6) will give different results.

5 Particular Cases

We can obtain the following generating functions for the Gauss's Hypergeometric functions.

I: Taking $n = 0$ in (4.7), we get

$$(1+t)^{c-1} {}_2F_1(a,b;c;x(1+t)) \tag{5.1}$$

$$= \sum_{m=0}^{\infty} \frac{(c-m)_m}{m!} \ {}_2F_1(a,b;c-m;x) t^m$$

II: Taking $n = 0$ in (4.10), we get

$$(1-y)^{a+b-c}{}_2F_1(a,b;c;(x-1)(1-y)+1) \tag{5.2}$$

$$= \sum_{k=0}^{\infty} \frac{(c-a)_k(c-b)_k}{(c)_k k!} \;\; {}_2F_1(a,b;c+k;x)y^k$$

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