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Lie Group Analysis of Conformable Fractional Kadamtsev-Petviashvili Equation

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Abstract:

Lie group analysis is used to investigate invariance properties of nonlinear fractional partial differential equations with three independent variables. The analysis is applied to the conformable fractional Kadamtsev-Petviashvili equation (KP). All of the vector fields and the Lie symmetries are obtained. We show that the conformable fractional KP equation can be converted to a conformable fractional partial differential equation with two independent variables. Henceforth, the produced equation can be transformed to a conformable fractional ordinary differential equation.

1 Introduction

Fractional calculus (FC) is regarded as a generalization of the classical differentiation and integration for arbitrary noninteger (real or complex) order. FC is almost as old as the classical calculus and goes back to times when Leibniz and Newton invented differential calculus. It has been developed progressively up to now, and it is regarded as a novel topic. After 1974, FC is rapidly developed. Fractional derivatives and integrals have many uses and they themselves have arisen from certain requirements in applications. Some of known fractional derivatives are Riemann-Liouville, modified Riemann-Liouville, Caputo, Hadmard, Erdélyi-Kober, Riesz, Grünwald-Letnikov, Marchaud and others; see [1]–[8].

In 2014, a new definition of fractional derivative with an interesting idea that extends the familiar limit definition of the derivatives of a function has been introduced by Khalil, Alhorani, Yousef, and Sababheh [9]. The new definition is called the conformable fractional derivative. Unlike other definitions, this definition prominently compatible with the classical derivative and it seems to satisfy all the requirements of the standard derivative. The importance of the conformable fractional derivative lies in satisfying the product and quotient formulas. Moreover, it has a simple formula for the chain rule. Since Khalil's definition, conformable fractional derivative has become more and more pervasive for its effectiveness on other mathematical disciplines [10]–[29].

The Lie symmetry theory is regarded as an important tool in studying the analysis of differential equations. Furthermore, it is the most effective method for constructing analytical solutions of nonlinear differential equations. Lie's theory is powerful, versatile, and fundamental to the development of systematic procedures leading to the integration by quadrature or at least to lowering the order of ordinary differential equations, to the determination of invariant solutions of initial and boundary value problems, to the derivation of conserved quantities, or to the construction of relations between different differential equations that turn out to be equivalent [30]-[33]. In the beginning of the 19^{th} century the first work in the subject of Lie symmetry was given by the Norwegian mathematician Sophus Lie. After that many studies have been devoted exclusively to the theory of the Lie symmetry groups and their applications to the field of differential equations [34]–[36]. Lie group analysis of fractional differential equations was investigated recently for example see [37]–[54]. Some of these studies were with Caputo fractional derivative [49], [50], modified Riemann-Liouville fractional derivative [54], and Riesz fractional derivative [53]. The most of the studies were with

Riemann-Liouville fractional derivative [37]–[48]. However, each of the equations under consideration with Riemann-Liouville fractional derivative can be transformed to a nonlinear ordinary fractional differential equations where the fractional derivative in the reduced equations is turned out to be the Erdélyi-Kober fractional derivative depending on a parameter β. However, the produced equations can not be solvable as in the classical derivative.

In [55] the generalized Lie symmetry technique is proposed for derivation of point symmetries of FPDE with two independent variables. The method is applied to Korteweg-de Vries, modified Korteweg-de Vries, Burgers, and modified Burgers equations with conformable fractional time-derivative and classical space-derivative. It was shown that with the help of derived Lie point symmetries the corresponding invariant solutions convert each of the considered equations to a nonlinear ordinary differential equation with classical derivative. Also in [56] the Lie symmetry analysis of Korteweg-de Vries, modified Korteweg-de Vries, Burgers and modified Burgers equations with conformable fractional time and space partial derivatives are established. For each equation all of the vector fields and the Lie symmetries are constructed. We show that these equations can be converted to either fractional ordinary differential equations with conformable derivative or ordinary differential equations with classical derivative.

In this work, the Lie symmetry analysis is developed to investigate nonlinear time-space fractional partial differential equations with three independent variables. This method which utilizes the development prolongation formulas is applied to time-space conformable fractional KP equation. All of the generators of the vector fields and some of the invariant variables are introduced. The KP equation can be transformed to an easier form of conformable fractional partial differential equation. Henceforward, the produced equation can be converted to a conformable fractional ordinary differential equation.

2 Conformable Fractional Calculus

Definition1. [9] Given a function $f:[0,\infty)\to\mathbb{R}$, the conformable fractional derivative of order α of f is defined by

$$D^{\alpha}[f(t)] = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon} \qquad (1)$$

for all $t>0$, $\alpha\in(0,1]$. If $D^{\alpha}[f(t)]$ exists for t in some interval $(0,a)$, $a>0$, and $\lim_{t\to 0^+} D^{\alpha}[f(t)]$ exists, then

$$D^{\alpha}[f(0)] = \lim_{t\to 0^+} D^{\alpha}[f(t)].$$

If $D^{\alpha}[f(t)]$ exists for $t\in[0,\infty)$, then f is said to be α-differentiable at t. One should notice that a function could be α-differentiable at a point but not differentiable at the same point. For instance, $f(t)=\sqrt{t}$, $D^{\frac{1}{2}}[f(t)]=\frac{1}{2}$. Consequently, $D^{\frac{1}{2}}[f](0)=\frac{1}{2}$, but the first derivative of f at $t=0$ does not exist.

Definition2. [9] Let $\alpha\in(n,n+1]$, and f be n differentiable at t, where $t>0$. Then the conformable fractional derivative of f of order α is defined by

$$D^{\alpha}f(t) = \lim_{\varepsilon\to 0}\frac{f^{(n)}(t+\varepsilon t^{n+1-\alpha})-f^{(n)}(t)}{\varepsilon}. \qquad (2)$$

Definition (2) can be rewritten in an equivalent form

$$[D^{\alpha}f](t) = [D^{\alpha-n}f^{(n)}](t).$$

With $\alpha\in(n,n+1]$ and f is $(n+1)$-differentiable at $t>0$, it is easily to show that

$$D^{\alpha}(f)(t) = t^{n+1-\alpha}f^{(n+1)}(t).$$

Theorem 1. [9] Let $\alpha \in (0,1]$ and f, g be α-differentiable at a point $t > 0$, then

1. $D^{\alpha}[af(t) + bg(t)] = a[D^{\alpha}f(t)] + b[D^{\alpha}g(t)]$, for all $a, b \in \mathbb{R}$.
2. If $f(t) = t^{p}$, then $D^{\alpha}[f(t)] = pt^{p-\alpha}$, for all $p \in \mathbb{R}$.
3. If $f(t) = \frac{t^{\alpha}}{\alpha}$, then $D^{\alpha}[f(t)] = 1$.
4. If f is the constant function defined by $f(t) = c$, then $D^{\alpha}[f(t)] = 0$.
5. $D^{\alpha}[f(t)g(t)] = f(t)D^{\alpha}[g(t)] + g(t)D^{\alpha}[f(t)]$.
6. $D^{\alpha}\left[\frac{f(t)}{g(t)}\right] = \frac{g(t)D^{\alpha}[f(t)] - f(t)D^{\alpha}[g(t)]}{[g(t)]^{2}}$.
7. If, in addition, f is differentiable, then $D^{\alpha}f(t) = t^{1-\alpha}\frac{d f(t)}{d t}$.

Definition 3. [9] Let $\alpha \in (n, n+1]$ then the conformable fractional derivative is given by $I^{\alpha}[f(t)] = I^{n+1}[t^{\alpha-n-1}f(t)] = \frac{1}{n!}\int_{0}^{t}\frac{(t-\tau)^{n}}{\tau^{1+n-\alpha}}f(\tau)d\tau$.

Definition 4. [9] $I^{\alpha}[f(t)] = I[t^{\alpha-1}f(t)] = \int_{0}^{t}\frac{f(\tau)}{\tau^{1-\alpha}}d\tau$, where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

Theorem 2. [9] $D^{\alpha}I^{\alpha}[f(t)] = f(t)$, for $t \geq 0$, where f is any Continuous function in the domain of I^{α}.

Theorem 3.[13] $I^{\alpha}D^{\alpha}[f(t)] = f(t) - \sum_{k=0}^{n}\frac{f^{(k)}(0)t^{k}}{k!}$, for $t \geq 0$, where f is an $(n+1)-$differentiable function, $n < \alpha \leq n+1$.

Lemma 1.[13] Let $f : [0, b) \rightarrow \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > 0$ we have $I^{\alpha}D^{\alpha}[f(t)] = f(t) - f(0)$.

Lemma 2.[16] Let $f : [0, b) \rightarrow \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > 0$ we have $D^{\alpha}D^{\beta}[f(t)] \neq D^{\alpha+\beta}[f(t)]$.

Lemma 3.[16] Let $0 < \alpha \leq 1$, f be α-differentiable at $g(t) > 0$, and g be α-differentiable at $t > 0$, then $D^{\alpha}[(f\, o g)(t)] = D^{\alpha}[f(g(t))]D^{\alpha}[g(t)][g(t)]^{\alpha-1}$.

Lemma 4.[23] Let $0 < \alpha \leq 1$, f be differentiable at $g(t)$, and g be α-differentiable at $t > 0$, then $D^{\alpha}[(f o g)(t)] = [f'(g(t))]D^{\alpha}[g(t)]$.

3 Lie Symmetry Analysis

In this section we will show that the Lie symmetry analysis can be successfully applied to time-space fractional partial differential equations with three independent variables where the fractional derivatives are in the conformable sense. Also we will propose a prolongation formula for the equations under consideration. We will deal with the following form of time-space fractional partial differential equations

$$\frac{\partial^{\beta}u}{\partial t^{\beta}} = F\left(t, x, y, u, \frac{\partial^{\alpha}u}{\partial x^{\alpha}}, \frac{\partial^{\gamma}u}{\partial y^{\gamma}}, \frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}, \frac{\partial^{2\gamma}u}{\partial y^{2\gamma}}, \dots\right), \quad 0 < \beta, \alpha, \gamma \leq 1, \qquad (3)$$

where $u = u(t, x, y)$, $F\left(t, x, y, u, \frac{\partial^{\alpha}u}{\partial x^{\alpha}}, \frac{\partial^{\gamma}u}{\partial y^{\gamma}}, \dots\right)$ is a nonlinear differential operator and $\frac{\partial^{\beta}u}{\partial t^{\beta}}$, $\frac{\partial^{\alpha}u}{\partial x^{\alpha}}$ and $\frac{\partial^{\gamma}u}{\partial y^{\gamma}}$ are the conformable fractional derivatives of order β, α and γ, respectively. Here $\frac{\partial^{n\alpha}u}{\partial x^{n\alpha}}$ are the sequential fractional derivatives given by

$$\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}} = \frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{\partial^{\alpha}u}{\partial x^{\alpha}}, \qquad \frac{\partial^{n\alpha}u}{\partial x^{n\alpha}} = \frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{\partial^{(n-1)\alpha}u}{\partial x^{(n-1)\alpha}}, \quad n = 3, 4, \dots \qquad (4)$$

The invertible one point transformations reads

$$\hat{t} = T(t, x, y, u, \varepsilon), \ \hat{x} = X(t, x, y, u, \varepsilon), \ \hat{y} = Y(t, x, y, u, \varepsilon), \ \hat{u} = U(t, x, y, u, \varepsilon). \quad (5)$$

In order to find a basis of the Lie group we need to create and then to solve the determining system of equations. The infinitesimal transformations of (5) are presented by

$$\hat{t} = t + \varepsilon\tau(t, x, y, u) + o(\varepsilon^2), \qquad (6a)$$

$$\hat{x} = x + \varepsilon\xi(t, x, y, u) + o(\varepsilon^2), \qquad (6b)$$

$$\hat{y} = y + \varepsilon\zeta(t, x, y, u) + o(\varepsilon^2), \qquad (6c)$$

$$\hat{u} = u + \varepsilon\eta(t, x, y, u) + o(\varepsilon^2). \qquad (6d)$$

It is adequate to introduce the infinitesimal operator

$$V = \tau(t, x, y, u)\frac{\partial}{\partial t} + \xi(t, x, y, u)\frac{\partial}{\partial x} + \zeta(t, x, y, u)\frac{\partial}{\partial y} + \eta(t, x, y, u)\frac{\partial}{\partial u}.$$

Now, the group transformations (5) corresponding to operator (7) can be obtained by solving the Lie equations

$$\frac{d\hat{t}}{d\varepsilon} = \tau(\hat{t}, \hat{x}, \hat{y}, \hat{u}), \quad \frac{d\hat{x}}{d\varepsilon} = \xi(\hat{t}, \hat{x}, \hat{y}, \hat{u}), \quad \frac{d\hat{y}}{d\varepsilon} = \zeta(\hat{t}, \hat{x}, \hat{y}, \hat{u}), \quad \frac{d\hat{u}}{d\varepsilon} = \eta(\hat{t}, \hat{x}, \hat{y}, \hat{u}), \quad (8)$$

subject to the initial conditions

$$\hat{t}|_{\varepsilon=0} = t, \quad \hat{x}|_{\varepsilon=0} = x, \quad \hat{y}|_{\varepsilon=0} = y, \quad \hat{u}|_{\varepsilon=0} = u.$$

In this case, a surface $u = u(t, x, y)$ is converted to it self by the group of transformations generated by V if

$$V\,(u - u(t, x, y)) = 0 \quad \text{whenever} \quad u = u(t, x, y). \qquad (9)$$

Now, the transformation (5) form a symmetry group G of equation (3) if the function $\hat{u}(\hat{t},\hat{x},\hat{y})$ satisfies the equation

$$\frac{\partial^{\beta}\hat{u}}{\partial\hat{t}^{\beta}} = F\left(\hat{t},\hat{x},\hat{y},\hat{u},\frac{\partial^{\alpha}\hat{u}}{\partial\hat{x}^{\alpha}},\frac{\partial^{\gamma}\hat{u}}{\partial\hat{y}^{\gamma}},\frac{\partial^{2\alpha}\hat{u}}{\partial\hat{x}^{2\alpha}},\frac{\partial^{2\gamma}\hat{u}}{\partial\hat{y}^{2\gamma}},\dots\right),\qquad 0<\beta,\ \alpha,\ \gamma\le 1, \tag{10}$$

whenever the function $u = u(t,x,y)$ satisfies equation (3). Extending transformation (5) to various kinds of derivatives, we can obtain

$$\frac{\partial^{\beta}\hat{u}}{\partial\hat{t}^{\beta}} = \frac{\partial^{\beta}u}{\partial t^{\beta}} + \varepsilon\eta^{t_\beta}(t,x,y,u,u_t,u_x,u_y) + O(\varepsilon^2), \tag{11a}$$

$$\frac{\partial^{\alpha}\hat{u}}{\partial\hat{x}^{\alpha}} = \frac{\partial^{\alpha}u}{\partial x^{\alpha}} + \varepsilon\eta^{x_\alpha}(t,x,y,u,u_t,u_x,u_y) + O(\varepsilon^2), \tag{11b}$$

$$\frac{\partial^{\gamma}\hat{u}}{\partial\hat{y}^{\gamma}} = \frac{\partial^{\gamma}u}{\partial y^{\gamma}} + \varepsilon\eta^{y_\gamma}(t,x,y,u,u_t,u_x,u_y) + O(\varepsilon^2), \tag{11c}$$

$$\frac{\partial^{2\alpha}\hat{u}}{\partial\hat{x}^{2\alpha}} = \frac{\partial^{2\alpha}u}{\partial x^{2\alpha}} + \varepsilon\eta^{x_\alpha x_\alpha}(t,x,y,u,u_t,u_x,u_y,\dots) + O(\varepsilon^2), \tag{11d}$$

$$\frac{\partial^{2\gamma}\hat{u}}{\partial\hat{y}^{2\gamma}} = \frac{\partial^{2\gamma}u}{\partial y^{2\gamma}} + \varepsilon\eta^{y_\gamma y_\gamma}(t,x,y,u,u_t,u_x,u_y,\dots) + O(\varepsilon^2), \tag{11e}$$

$$\frac{\partial^{3\alpha}\hat{u}}{\partial\hat{x}^{3\alpha}} = \frac{\partial^{3\alpha}u}{\partial x^{3\alpha}} + \varepsilon\eta^{x_\alpha x_\alpha x_\alpha}(t,x,y,u,u_t,u_x,u_y,\dots) + O(\varepsilon^2), \tag{11f}$$

$$\frac{\partial^{3\gamma}\hat{u}}{\partial\hat{y}^{3\gamma}} = \frac{\partial^{3\gamma}u}{\partial y^{3\gamma}} + \varepsilon\eta^{y_\gamma y_\gamma y_\gamma}(t,x,y,u,u_t,u_x,u_y,\dots) + O(\varepsilon^2), \tag{11g}$$

where

$$\begin{aligned}
\eta^{t_\beta} &= t^{1-\beta}\eta^t + (1-\beta)t^{-\beta}\tau u_t\,,\\
\eta^{x_\alpha} &= x^{1-\alpha}\eta^x + (1-\alpha)x^{-\alpha}\xi u_x\,,\\
\eta^{y_\gamma} &= y^{1-\gamma}\eta^y + (1-\gamma)y^{-\gamma}\zeta u_y\,,\\
\eta^{x_\alpha x_\alpha} &= x^{2-2\alpha}\eta^{xx} + (1-\alpha)x^{1-2\alpha}\eta^x + 2(1-\alpha)x^{1-2\alpha}\xi u_{xx}\\
&\quad + (1-\alpha)(1-2\alpha)x^{-2\alpha}\xi u_x,\\
\eta^{y_\gamma y_\gamma} &= y^{2-2\gamma}\eta^{yy} + (1-\gamma)y^{1-2\gamma}\eta^y + 2(1-\gamma)y^{1-2\gamma}\zeta u_{yy}\\
&\quad +(1-\gamma)(1-2\gamma)y^{-2\gamma}\zeta u_y,\\
\eta^{x_\alpha x_\alpha x_\alpha} &= x^{3-3\alpha}\eta^{xxx} + 3(1-\alpha)x^{2-3\alpha}\eta^{xx} + (1-\alpha)(1-2\alpha)x^{1-3\alpha}\eta^x\\
&\quad + 3(1-\alpha)x^{2-3\alpha}\xi u_{xxx} + 3(1-\alpha)(2-3\alpha)x^{1-3\alpha}\xi u_{xx}\\
&\quad + (1-\alpha)(1-2\alpha)(1-3\alpha)x^{-3\alpha}\xi u_x,\\
\eta^{y_\gamma y_\gamma y_\gamma} &= y^{3-3\gamma}\eta^{yyy} + 3(1-\gamma)y^{2-3\gamma}\eta^{yy} + (1-\gamma)(1-2\gamma)y^{1-3\gamma}\eta^{\gamma}\\
&\quad + 3(1-\gamma)y^{2-3\gamma}\zeta u_{yyy} + 3(1-\gamma)(2-3\gamma)y^{1-3\gamma}\zeta u_{yy}\\
&\quad + (1-\gamma)(1-2\gamma)(1-3\gamma)y^{-3\gamma}\zeta u_y,\\
&\ \ \vdots
\end{aligned} \tag{12}$$

and

$$\eta^t = D_t(\eta) - u_tD_t(\tau) - u_xD_t(\xi) - u_yD_t(\zeta), \tag{13a}$$

$$\eta^x = D_x(\eta) - u_tD_x(\tau) - u_xD_x(\xi) - u_yD_x(\zeta), \tag{13b}$$

$$\eta^y = D_y(\eta) - u_tD_y(\tau) - u_xD_y(\xi) - u_yD_y(\zeta), \tag{13c}$$

$$\eta^{xx} = D_x(\eta^x) - u_{xt}D_x(\tau) - u_{xx}D_x(\xi) - u_{xy}D_x(\zeta), \tag{13d}$$

$$\eta^{yy} = D_y(\eta^y) - u_{yt}D_y(\tau) - u_{yx}D_y(\xi) - u_{yy}D_y(\zeta), \tag{13e}$$

$$\eta^{xxx} = D_x(\eta^{xx}) - u_{xxt}D_x(\tau) - u_{xxx}D_x(\xi) - u_{xxy}D_x(\zeta), \tag{13f}$$

$$\eta^{yyy} = D_y(\eta^{yy}) - u_{yyt}D_y(\tau) - u_{yyx}D_y(\xi) - u_{yyy}D_y(\zeta). \tag{13g}$$

$$\vdots$$

Here D_t, D_x and D_y denote the total derivative operators and are defined as

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{u_x} + u_{tt} \frac{\partial}{\partial u_t} + u_{yt} \frac{\partial}{\partial u_y} \cdots$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{u_x} + u_{tx} \frac{\partial}{\partial u_t} + u_{yx} \frac{\partial}{\partial u_y} \cdots$$

$$D_y = \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy} \frac{\partial}{u_x} + u_{ty} \frac{\partial}{\partial u_t} + u_{yy} \frac{\partial}{\partial u_y} \cdots$$

If the vector field (7) generates a symmetry of (3), then V must satisfy Lie symmetry condition

$$Pr^{(n\alpha, m\gamma, \beta)} V(\Delta_1)\big|_{\Delta_1=0} = 0, \tag{14}$$

where

$$\Delta_1 = \frac{\partial^\beta u}{\partial t^\beta} - F\left(t, x, y, u, \frac{\partial^\alpha u}{\partial x^\alpha}, \frac{\partial^\gamma u}{\partial y^\gamma}, \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}, \frac{\partial^{2\gamma} u}{\partial y^{2\gamma}}, \dots\right).$$

1 Time-space fractional Kadamtsev-Petviashvili equation

The following form of time-space fractional Kadamtsev-Petviashvili (KP) equation will be treated

$$\frac{\partial^\beta}{\partial t^\beta} \frac{\partial^\alpha u}{\partial x^\alpha} + a\left(\frac{\partial^\alpha u}{\partial x^\alpha}\right)^2 + au \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + b \frac{\partial^{4\alpha} u}{\partial x^{4\alpha}} + c \frac{\partial^{2\gamma} u}{\partial y^{2\gamma}} = 0, \tag{15}$$

where β, α and γ $(0 < \beta, \alpha, \gamma \leq 1)$ are parameters characterizing the orders of the conformable fractional time and space derivatives. Coinciding with the theory of Lie symmetry, one can achieve the infinitesimal criterion (14) of (15) by applying the $(4\alpha, 2\gamma, \beta)$ −prolongation $Pr^{(4\alpha, 2\gamma, \beta)} V$ to (15). The usual simplifications lead to the following relation

$$a\eta \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + 2a\eta^{x_\alpha} \frac{\partial^\alpha u}{\partial x^\alpha} + \eta^{x_\alpha t_\beta} + au\eta^{x_\alpha x_\alpha} + c\eta^{y_\gamma y_\gamma} + b\eta^{x_\alpha x_\alpha x_\alpha x_\alpha} = 0. \tag{16}$$

It is convenient to rewrite equation (16) in the form

$$a x^{2-2\alpha} \eta u_{xx} + a(1-\alpha) x^{1-2\alpha} \eta u_x + 2a x^{1-\alpha} \eta^{x_\alpha} u_x + \eta^{x_\alpha t_\beta} + au\eta^{x_\alpha x_\alpha}$$
$$+ c\eta^{y_\gamma y_\gamma} + b\eta^{x_\alpha x_\alpha x_\alpha x_\alpha} = 0\,, \tag{17}$$

which must be satisfied whenever equation (15) holds. Equation (15) simplifies at once to the form

$$x^{1-\alpha} t^{1-\beta} u_{xt} + a x^{2-2\alpha} u_x^2 + a x^{2-2\alpha} u u_{xx} + a(1-\alpha) x^{1-2\alpha} u u_x + c y^{2-2\gamma} u_{yy} +$$
$$c(1-\gamma) y^{1-2\gamma} u_y + b x^{4-4\alpha} u_{xxxx} + 6b(1-\alpha) x^{3-4\alpha} u_{xxx} +$$
$$b(1-\alpha)(7-11\alpha) x^{2-4\alpha} u_{xx} + b(1-\alpha)(1-2\alpha)(1-3\alpha) x^{1-4\alpha} u_x = 0. \tag{18}$$

Substituting the formulas for $\eta^{x_\alpha}, \eta^{x_\alpha x_\alpha}, \eta^{y_\gamma y_\gamma}, \eta^{x_\alpha x_\alpha x_\alpha x_\alpha}$ from equations (12, 13) and $\eta^{t_\beta x_\alpha} = t^{1-\beta}x^{1-\alpha}\eta^{tx} + (1-\beta)t^{-\beta}x^{1-\alpha}\tau u_{tx} + (1-\alpha)t^{1-\beta}x^{-\alpha}\xi u_{tx}$ with $\eta^{xt} = D_t(\eta^x) - u_{xt}D_t(\tau) - u_{xx}D_t(\xi) - u_{xy}D_t(\zeta)$ into equation (16). Replacing u_{xxxx} by $\frac{-1}{b}x^{3\alpha-3}t^{1-\beta}u_{xt} - \frac{a}{b}x^{2\alpha-2}u_x^2 - \frac{a}{b}x^{2\alpha-2}uu_{xx} - \frac{a}{b}(1-\alpha)x^{2\alpha-3}uu_x$ $-\frac{c}{b}x^{4\alpha-4}y^{2-2\gamma}u_{yy} - \frac{c}{b}(1-\gamma)x^{4\alpha-4}y^{1-2\gamma}u_y - 6(1-\alpha)x^{-1}u_{xxx} - (1-\alpha)(7-11\alpha)$ $x^{-2}u_{xx} - (1-\alpha)(1-2\alpha)(1-3\alpha)x^{-3}u_x$ whenever it occurs, and equating the coefficients of several monomials in partial derivatives of u, we obtain the full determining equations for the symmetry group of equation (15). The usual simplification for the full determining equations lead to the equations

$$\tau_u = \tau_x = \tau_y = \xi_u = \zeta_u = \zeta_x = \eta_{uu} = 0,$$
$$3x^{1-\alpha}t^{1-\beta}\xi_x - 3(1-\alpha)x^{-\alpha}t^{1-\beta}\xi + (1-\beta)x^{1-\alpha}t^{-\beta}\tau - x^{1-\alpha}t^{1-\beta}\tau_t = 0,$$
$$2cy^{2-2\gamma}\xi_y + x^{1-\alpha}t^{1-\beta}\zeta_t = 0,$$
$$4c(1-\gamma)y^{1-2\gamma}\xi_x - 4c(1-\alpha)(1-\gamma)x^{-1}y^{1-2\gamma}\xi + c(1-\gamma)(1-2\gamma)y^{-2\gamma}\zeta - cy^{2-2\gamma}\zeta_{yy} + 2cy^{2-2\gamma}\eta_{uy} - c(1-\gamma)y^{1-2\gamma}\zeta_y = 0,$$
$$2ax^{2-2\alpha}\xi_x - 2a(1-\alpha)x^{1-2\alpha}\xi + ax^{2-2\alpha}\eta_u = 0,$$
$$-6bx^{4-4\alpha}\xi_{xx} + 6b(1-\alpha)x^{3-4\alpha}\xi_x - 6b(1-\alpha)x^{2-4\alpha}\xi + 4bx^{4-4\alpha}\eta_{xu} = 0,$$
$$4cy^{2-2\gamma}\xi_x - 4c(1-\alpha)x^{-1}y^{2-2\gamma}\xi + 2c(1-\gamma)y^{1-2\gamma}\zeta - 2cy^{2-2\gamma}\zeta_y = 0,$$
$$3b(1-\alpha)(1-2\alpha)(1-3\alpha)x^{1-4\alpha}\xi_x + 3a(1-\alpha)x^{1-2\alpha}u\xi_x - bx^{4-4\alpha}\xi_{xxxx} - ax^{2-2\alpha}u\xi_{xx} - 6b(1-\alpha)x^{3-4\alpha}\xi_{xxx} - b(1-\alpha)(7-11\alpha)x^{2-4\alpha}\xi_{xx} - 3b(1-\alpha)(1-2\alpha)(1-3\alpha)x^{-4\alpha}\xi - a(3-2\alpha)(1-\alpha)x^{-2\alpha}u\xi - cy^{2-2\gamma}\xi_{yy} - c(1-\gamma)y^{1-2\gamma}\xi_y - x^{1-\alpha}t^{1-\beta}\xi_{xt} + x^{1-\alpha}t^{1-\beta}\eta_{ut} + 2ax^{2-2\alpha}\eta_x + a(1-\alpha)x^{1-2\alpha}\eta = 0,$$
$$2b(1-\alpha)(7-11\alpha)x^{2-4\alpha}\xi_x + 2ax^{2-2\alpha}u\xi_x - 4bx^{4-4\alpha}\xi_{xxx} - 18b(1-\alpha)x^{3-4\alpha}\xi_{xx} - 2b(1-\alpha)(7-11\alpha)x^{1-4\alpha}\xi - 2a(1-\alpha)x^{1-2\alpha}u\xi - x^{1-\alpha}t^{1-\beta}\xi_t + ax^{2-2\alpha}\eta = 0,$$
$$x^{1-\alpha}t^{1-\beta}\eta_{xt} + bx^{4-4\alpha}\eta_{xxxx} + 6b(1-\alpha)x^{3-4\alpha}\eta_{xxx} + b(1-\alpha)(7-11\alpha)x^{2-4\alpha}\eta_{xx} + b(1-\alpha)(1-2\alpha)(1-3\alpha)x^{1-4\alpha}\eta_x + ax^{2-2\alpha}u\eta_{xx} + a(1-\alpha)x^{1-2\alpha}u\eta_x + cy^{2-2\gamma}\eta_{yy} + c(1-\gamma)y^{1-2\gamma}\eta_y = 0. \quad (19)$$

The solution of the system of equations (19) is elementary and is given by

$$\tau = \frac{c_1}{\beta}t + c_2t^{1-\beta}, \quad (20a)$$
$$\xi = \frac{c_1}{3\alpha}x - \frac{1}{2c\gamma}y^\gamma x^{1-\alpha}t^{1-\beta}\psi'(t) + x^{1-\alpha}\phi(t), \quad (20b)$$
$$\zeta = \frac{2c_1}{3\gamma}y + y^{1-\gamma}\psi(t), \quad (20c)$$
$$\eta = \frac{-2c_1}{3}u - \frac{(1-\beta)}{2ac\gamma}y^\gamma t^{1-2\beta}\psi'(t) - \frac{1}{2ac\gamma}y^\gamma t^{2-2\beta}\psi''(t) + \frac{1}{a}t^{1-\beta}\phi'(t). \quad (20d)$$

As a result, the Lie group generators are given by

$$V_1 = t^{1-\beta}\frac{\partial}{\partial t}, \tag{21a}$$

$$V_2 = \frac{t}{\beta}\frac{\partial}{\partial t} + \frac{x}{3\alpha}\frac{\partial}{\partial x} + \frac{2y}{3\gamma}\frac{\partial}{\partial y} - \frac{2u}{3}\frac{\partial}{\partial u}, \tag{21b}$$

$$V_3 = x^{1-\alpha}\phi(t)\ \frac{\partial}{\partial x} + \frac{1}{a}t^{1-\beta}\phi'(t)\frac{\partial}{\partial u}, \tag{21c}$$

$$V_4 = \frac{-1}{2c\gamma}y^{\gamma}x^{1-\alpha}t^{1-\beta}\psi'(t)\frac{\partial}{\partial x} + y^{1-\gamma}\psi(t)\frac{\partial}{\partial y} - \frac{(1-\beta)}{2ac\gamma}y^{\gamma}t^{1-2\beta}\psi'(t)\frac{\partial}{\partial u} - \frac{1}{2ac}y^{\gamma}t^{2-2\beta}\psi''\frac{\partial}{\partial u}. \tag{21}$$

It is worth noting that the Lie algebra for fractional and classical KP equation have the same dimension and when $\beta = \alpha = \gamma = 1$ the Lie algebra for the fractional KP equation reduces to that of classical KP equation [36]. Now we will check the closenesses property of the vector fields under the Lie bracket to satisfy this condition we need to put further constraints on the functions $\psi(t)$ and $\phi(t)$. So let us set $\kappa(t) = 0$, and $\vartheta = 0$. As a result we have

$$V_3 = 0, \quad V_4 = 0, \quad \text{whereas} \quad [V_1, V_2] = V_1.$$

Moreover, the symmetry algebra of equation (15) is spanned by the two vector fields

$$V_1 = t^{1-\beta}\frac{\partial}{\partial t}, \qquad V_2 = \frac{t}{\beta}\frac{\partial}{\partial t} + \frac{x}{3}\frac{\partial}{\partial x} + \frac{2y}{3}\frac{\partial}{\partial y} - \frac{2u}{3}\frac{\partial}{\partial u}. \tag{22}$$

Employing the infinitesimal generator V_2 the relevant similarity variables can be found by solving the characteristic equation

$$\frac{\beta\, dt}{3t} = \frac{\alpha\, dx}{x} = \frac{\gamma\, dy}{2y} = \frac{-du}{2u}. \tag{23}$$

Consequently, the associated invariance are determined by

$$v = xt^{\frac{-\beta}{3\alpha}}, \qquad \omega = yt^{\frac{-2\beta}{3\gamma}}, \qquad u = \Psi(v,\omega)t^{\frac{-2\beta}{3}}. \tag{24}$$

The transformation (24) convert the differential equation (15) into

$$b\frac{\partial^{4\alpha}\Psi}{\partial v^{4\alpha}} + a\Psi\frac{\partial^{2\alpha}\Psi}{\partial v^{2\alpha}} + c\frac{\partial^{2\gamma}\Psi}{\partial \omega^{2\gamma}} + a\left(\frac{\partial^{\alpha}\Psi}{\partial v^{\alpha}}\right)^2 - \frac{\beta}{3}\left(\frac{1}{\alpha}+2\right)\frac{\partial^{\alpha}\Psi}{\partial v^{\alpha}} + v\frac{\partial^{\alpha}}{\partial v^{\alpha}}\frac{\partial\Psi}{\partial v} + \omega\frac{\partial^{\alpha}}{\partial v^{\alpha}}\frac{\partial\Psi}{\partial\omega} = 0. \tag{25}$$

Equation (25) is a nonlinear fractional partial differential equation with conformable sense and two independent variables. It is convenient to switch equation (15) to ordinary differential equation. To achieve this goal we investigate the Lie symmetry analysis for the introduced equation (25). Applying the $(4\alpha, 2\gamma)-$prolongation $Pr^{(4\alpha,2\gamma)}V$ to (25) the infinitesimal criterion is given by

$$\xi\frac{\partial^{\alpha}}{\partial v^{\alpha}}\frac{\partial\Psi}{\partial v} + \zeta\frac{\partial^{\alpha}}{\partial v^{\alpha}}\frac{\partial\Psi}{\partial\omega} + a\eta\frac{\partial^{2\alpha}\Psi}{\partial v^{2\alpha}} + 2a\eta^{v_\alpha}\frac{\partial^{\alpha}\Psi}{\partial v^{\alpha}} - \frac{\beta}{3}\left(\frac{1}{\alpha}+2\right)\eta^{v_\alpha} + a\eta^{v_\alpha v_\alpha}\Psi + v\eta^{v_\alpha v} + \omega\eta^{v_\alpha\omega} + c\eta^{\omega_\gamma\omega_\gamma} + b\eta^{v_\alpha v_\alpha v_\alpha v_\alpha} = 0, \tag{26}$$

which may be written as

$$\xi \upsilon^{1-\alpha}\Psi_{\upsilon\upsilon} + \zeta \upsilon^{1-\alpha}\Psi_{\upsilon\omega} + a\eta \upsilon^{2-2\alpha}\Psi_{\upsilon\upsilon} + a\eta(1-\alpha)\upsilon^{1-2\alpha}\Psi_{\upsilon} + 2a\eta^{\upsilon_\alpha}\upsilon^{1-\alpha}\Psi_{\upsilon} -$$

$$\frac{\beta}{3}\left(\frac{1}{\alpha}+2\right)\eta^{\upsilon_\alpha} + a\eta^{\upsilon_\alpha\upsilon_\alpha}\Psi + \upsilon\eta^{\upsilon_\alpha\upsilon} + \omega\eta^{\upsilon_\alpha\omega} + c\eta^{\omega_\gamma\omega_\gamma} + b\eta^{\upsilon_\alpha\upsilon_\alpha\upsilon_\alpha\upsilon_\alpha} = 0. \tag{27}$$

Where the formal expansions of η^{υ_α}, $\eta^{\upsilon_\alpha\upsilon_\alpha}$, $\eta^{\omega_\gamma\omega_\gamma}$, $\eta^{\upsilon_\alpha\upsilon}$, $\eta^{\upsilon_\alpha\omega}$, and $\eta^{\upsilon_\alpha\upsilon_\alpha\upsilon_\alpha\upsilon_\alpha}$ are given by

$$\eta^{\upsilon_\alpha} = \upsilon^{1-\alpha}\eta^{\upsilon} + (1-\alpha)\upsilon^{-\alpha}\xi\Psi_{\upsilon},$$

$$\eta^{\upsilon_\alpha\upsilon_\alpha} = \upsilon^{2-2\alpha}\eta^{\upsilon\upsilon} + (1-\alpha)\upsilon^{1-2\alpha}\eta^{\upsilon} + 2(1-\alpha)\upsilon^{1-2\alpha}\xi\Psi_{\upsilon\upsilon} + (1-\alpha)(1-2\alpha)\upsilon^{-2\alpha}\xi\Psi_{\upsilon},$$

$$\eta^{\omega_\gamma\omega_\gamma} = \omega^{2-2\gamma}\eta^{\omega\omega} + (1-\gamma)\omega^{1-2\gamma}\eta^{\omega} + 2(1-\gamma)\omega^{1-2\gamma}\zeta\Psi_{\omega\omega} + (1-\gamma)(1-2\gamma)\omega^{-2\gamma}\zeta\Psi_{\omega},$$

$$\eta^{\upsilon_\alpha\upsilon} = \upsilon^{1-\alpha}\eta^{\upsilon\upsilon} + (1-\alpha)\upsilon^{-\alpha}\xi\Psi_{\upsilon\upsilon},$$

$$\eta^{\upsilon_\alpha\upsilon} = \upsilon^{1-\alpha}\eta^{\upsilon\omega} + (1-\alpha)\upsilon^{-\alpha}\xi\Psi_{\upsilon\omega},$$

$$\eta^{\upsilon_\alpha\upsilon_\alpha\upsilon_\alpha\upsilon_\alpha} = \upsilon^{4-4\alpha}\eta^{\upsilon\upsilon\upsilon\upsilon} + 6(1-\alpha)\upsilon^{3-4\alpha}\eta^{\upsilon\upsilon\upsilon} + (1-\alpha)(7-11\alpha)\upsilon^{2-4\alpha}\eta^{\upsilon\upsilon} + (1-\alpha)(1-2\alpha)(1-3\alpha)\upsilon^{1-4\alpha}\eta^{\upsilon} + 4(1-\alpha)\upsilon^{3-4\alpha}\xi\Psi_{\upsilon\upsilon\upsilon\upsilon} + 6(1-\alpha)(3-4\alpha)\upsilon^{2-4\alpha}\xi\Psi_{\upsilon\upsilon\upsilon} + (1-\alpha)(7-11\alpha)(2-4\alpha)\upsilon^{1-4\alpha}\xi\Psi_{\upsilon\upsilon} + (1-\alpha)(1-2\alpha)(1-3\alpha)(1-4\alpha)\upsilon^{-4\alpha}\xi\Psi_{\upsilon}. \tag{28}$$

The conventionally treatment of the equation (27) leads to the following full determining equations for the symmetry group of equation (25)

$$\zeta_\Psi = \zeta_\upsilon = \xi_\Psi = \eta_{\Psi\Psi} = 0,$$

$$\eta_\Psi + 2\xi_\upsilon - 2(1-\alpha)\upsilon^{-1}\xi = 0,$$

$$2\xi_\upsilon - 2(1-\alpha)\upsilon^{-1}\xi - \zeta_\omega + (1-\gamma)\omega^{-1}\zeta = 0,$$

$$6b(1-\alpha)\xi_\upsilon + 4b\upsilon\eta_{\Psi\upsilon} - 6b\upsilon\xi_{\upsilon\upsilon} - 6(1-\alpha)b\upsilon^{-1}\xi = 0,$$

$$3\xi_\upsilon - 3(1-\alpha)\upsilon^{-1}\xi - 2c\omega^{1-2\gamma}\upsilon^{\alpha-1}\xi_\omega - \zeta_\omega + \omega^{-1}\zeta = 0,$$

$$4\xi_\upsilon - 4(1-\alpha)\upsilon^{-1}\xi + \frac{2}{(1-\gamma)}\omega\eta_{\Psi\omega} - \frac{1}{(1-\gamma)}\omega\zeta_{\omega\omega} - \zeta_\omega + (1-2\gamma)\omega^{-1}\zeta + \frac{1}{c}\omega^{2\gamma}\upsilon^{1-\alpha}\eta_{\Psi\upsilon} = 0,$$

$$2\upsilon^{2-\alpha}\xi_\upsilon + 2a\upsilon^{2-2\alpha}\Psi\xi_\upsilon + 2b(1-\alpha)(7-11\alpha)\upsilon^{2-4\alpha}\xi_\upsilon + 6b\upsilon^{4-4\alpha}\eta_{\upsilon\upsilon\Psi} - 4b\upsilon^{4-4\alpha}\xi_{\upsilon\upsilon\upsilon} + a\upsilon^{2-2\alpha}\eta + 18b(1-\alpha)\upsilon^{1-4\alpha}\eta_{\upsilon\Psi} - 18b(1-\alpha)\upsilon^{3-4\alpha}\xi_{\upsilon\upsilon} - 2b(1-\alpha)(7-11\alpha)\upsilon^{1-4\alpha}\xi + (-2+3\alpha)\upsilon^{1-\alpha}\xi - \omega\upsilon^{1-\alpha}\xi_\omega - 2a(1-\alpha)\upsilon^{1-2\alpha}\Psi\xi = 0,$$

$3a(1-\alpha)\upsilon^{1-2\alpha}\Psi\xi_\upsilon - 4a(1-\alpha)^2\upsilon^{-2\alpha}\Psi\xi - a\upsilon^{2-2\alpha}\Psi\xi_{\upsilon\upsilon} +$
$a(1-\alpha)(1-2\alpha)\upsilon^{-2\alpha}\Psi\xi - \beta\left(\frac{1}{\alpha}+2\right)\upsilon^{1-\alpha}\xi_\upsilon - \omega\upsilon^{1-\alpha}\xi_{\upsilon\omega} +$
$3(1-\alpha)(1-2\alpha)(1-3\alpha)b\upsilon^{1-4\alpha}\xi_\upsilon - b\upsilon^{4-4\alpha}\xi_{\upsilon\upsilon\upsilon\upsilon} -$
$6(1-\alpha)b\upsilon^{3-4\alpha}\xi_{\upsilon\upsilon\upsilon} - (1-\alpha)(7-11\alpha)b\upsilon^{2-2\alpha}\xi_{\upsilon\upsilon} -$
$3(1-\alpha)(1-2\alpha)(1-3\alpha)b\upsilon^{-4\alpha}\xi - c(1-\gamma)\omega^{1-2\gamma}\xi_\omega +$
$\beta(1-\alpha)\left(\frac{1}{\alpha}+2\right)\upsilon^{-\alpha}\xi - c\omega^{2-2\gamma}\xi_{\omega\omega} - \upsilon^{2-\alpha}\xi_{\upsilon\upsilon} + \omega\upsilon^{1-\alpha}\eta_{\omega\Psi} +$
$2a\upsilon^{2-2\alpha}\eta_\upsilon + a(1-\alpha)\upsilon^{1-2\alpha}\eta = 0,$

$b\upsilon^{4-4\alpha}\eta_{\upsilon\upsilon\upsilon\upsilon} + 6b(1-\alpha)\upsilon^{3-4\alpha}\eta_{\upsilon\upsilon\upsilon} + b(1-\alpha)(7-11\alpha)\upsilon^{2-4\alpha}\eta_{\upsilon\upsilon} +$
$\omega\upsilon^{1-\alpha}\eta_{\upsilon\omega} + b(1-\alpha)(1-2\alpha)(1-3\alpha)\upsilon^{1-4\alpha}\eta_\upsilon + c\omega^{2-2\gamma}\eta_{\omega\omega} +$
$c(1-\gamma)\omega^{1-2\gamma}\eta_\omega + \upsilon^{2-\alpha}\eta_{\upsilon\upsilon} + a\upsilon^{2-2\alpha}\Psi\eta_{\upsilon\upsilon} + a(1-\alpha)\upsilon^{1-2\alpha}\Psi\eta_\upsilon -$
$\frac{\beta}{3}\left(\frac{1}{\alpha}+2\right)\upsilon^{1-\alpha}\eta_\upsilon = 0.$ (29)

The usual simplification for the system of equations (29) leads to

$$\zeta = c_1\,\omega^{1-\gamma}, \quad \xi = \frac{c_1}{2c}\omega^\gamma\upsilon^{1-\alpha} + c_2\upsilon^{1-\alpha}, \quad \eta = \frac{-c_1}{2ac}(\alpha-\gamma)\omega^\gamma - \frac{\alpha}{a}c_2, \quad (30)$$

where c_1 and c_2 are arbitrary constants. Hence, the symmetry group of equation (25) is spanned by the two vector fields

$$V_1 = \upsilon^{1-\alpha}\frac{\partial}{\partial\upsilon} - \frac{\alpha}{a}\frac{\partial}{\partial\Psi}, \quad V_2 = \omega^{1-\gamma}\frac{\partial}{\partial\omega} + \frac{1}{2c}\omega^\gamma\upsilon^{1-\alpha}\frac{\partial}{\partial\upsilon} - \frac{(\alpha-\gamma)}{2ac}\omega^\gamma\frac{\partial}{\partial\Psi}. \quad (31)$$

Moreover, the Lie group is closed under the Lie bracket, that is;

$$[V_1, V_2] = 0. \quad (32)$$

On the other hand, the similarity variables for the infinitesimal generator V_1 can be approached by solving the relative characteristic equation

$$\frac{d\upsilon}{\upsilon^{1-\alpha}} = \frac{-a\,d\Psi}{\alpha}, \quad (33)$$

which implies the invariant variables

$$\Psi = \frac{-1}{a}\upsilon^\alpha + \Phi(\omega). \quad (34)$$

The successive application of transform (34) into equation (25) leads to the second order linear ordinary differential equation with classical sense

$$\Phi''(\omega) + (1-\gamma)\omega^{-1}\Phi'(\omega) = \frac{-\alpha}{ac}\left[\frac{\beta}{3\alpha} + \frac{2\beta}{3} + 1\right]\omega^{2\gamma-2}. \quad (35)$$

Hence, the solution $\Phi(\omega)$ is given by

$$\Phi(\omega) = \frac{-\alpha}{2ac\gamma^2}\left[\frac{\beta}{3\alpha} + \frac{2\beta}{3} + 1\right]\omega^{2\gamma} + \frac{\delta_1}{\gamma}\omega^\gamma + \delta_2, \quad (36)$$

where δ_1, δ_2 are constants of integration.

In the remainder of this section we will find the similarity variable for the infinitesimal generator V_2, which can be introduced by using a similar consideration. Solving the associate characteristic equation

$$\frac{d\omega}{\omega^{1-\gamma}} = \frac{2cd\upsilon}{\omega^{\gamma}\upsilon^{1-\alpha}} = \frac{-2acd\Psi}{(\alpha-\gamma)\omega^{\gamma}} \qquad (37)$$

It follows immediately that

$$\varpi = \frac{2c}{\alpha}\upsilon^{\alpha} - \frac{1}{2\gamma}\omega^{2\gamma}, \qquad \Psi = \frac{(\gamma-\alpha)}{4ac\gamma}\omega^{2\gamma} + \Theta(\varpi). \quad (38)$$

Substituting the transformation (38) into the differential equation (25) yields

$$16c^4 b\Theta'''' + 4c^2 a\Theta\Theta'' + 2c\alpha\varpi\Theta'' + 4c^2 a\Theta'^2 + c\left[-\gamma + 2\alpha - 2 - \frac{2\beta}{3}\left(\frac{1}{\alpha} + 2\right)\right]\Theta' = \frac{-\gamma(\gamma-\alpha)}{2a}. \qquad (39)$$

It is worth mentioning that the results in the time-fractional KP equation and the time-space fractional KP equation are consistence and they are the same when we substitute $\alpha = \gamma = 1$ in the time-space fractional KP. Also they are consistence with the classical KP equation and have the same results when $\beta = \alpha = \gamma = 1$.

5 Conclusion

In summary, we have applied the theory of Lie symmetry with three independent variables. The methodology is used to investigate invariance properties of nonlinear time-space conformable fractional partial differential equations with three independent variables.

The efficiency of the method is illustrated by its application to the conformable time-space fractional Kadamtsev-Petviashvili equation. Subsequently, the relevant vector fields and the symmetry reductions have been derived. It should be noted that the similarity reduction method converts the equation under consideration to an easier form of partial differential equation with conformable fractional derivatives, moreover, the number of the independent variables is reduced. Henceforward, the new equation has been converted to an ordinary conformable fractional differential equation. It is evident that any solution for the produced equation is also a solution of the original equation.

It is worth mentioning that the similarity reduction method convert a conformable time-fractional partial differential equation with more than two independent variables to a partial differential equation with classical derivative. However, the methodology transformed the time-space conformable fractional partial differential equation with more than two independent variables to a fractional partial differential equation with conformable concept.

As a future work, one can perform the Lie symmetry analysis to other partial differential equations with time and time-space fractional derivatives. Moreover, it may be interesting to apply the same methodology for systems of time and time-space fractional differential equations whenever the time and space derivatives are the conformable fractional derivatives.

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