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A New Fractionally Conformable Form of Hilbert's Integral Inequality

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حول شكل كسري توافقي جديد لمتباينات هلبرت التكاملية

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Abstract:

Fractional integral of non-integer order is considered as one of the most important topics dealt with in scientific research during the past decades, and has evolved significantly in past years. We employ one of the recent fractional calculus(the conformable one) and derive some new integral inequalities related to Hilbert's type inequalities. The best possible constant is determined for general and some other special cases.

Keywords: Conformable derivative ; Conformable integral; Hilbert's inequality; Hardy-Hilbert's inequality; Best possible constant.

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الملخص:

تعتبر التكاملات الكسرية من أهم الموضوعات التي تم تناولها في مجال البحث العلمي خلال العقود الماضية ، وقد تطورت بشكل كبير في السنوات الأخيرة.

نتناول في هذا البحث بالدراسة نوعاً جديداً من حساب التفاضل والتكامل الكسري (التوافقي) ونشتق أنواع جديدة من المتباينات التكاملية من نوع هلبرت .

كما نقوم بحساب أفضل ثابت ممكن في الحالات العامة ولبعض الحالات الخاصة الأخرى.

Introduction

The conformable derivative and conformable integral came as a new Fractional Calculus that introduced and studied by some mathematicians in the recent years. In [Khalil *et. al*. 2014], the author gave the definitions of conformable derivative and conformable integral as follows:

Definition 1.3 (Conformable derivative) [Khalil $et.al.$ 2014]
Let $f:[0,\infty)\to\mathbb{R}$ and $t>0$. Then the Conformable derivative of f of order α is defined by

$$D^{\alpha}(f)(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon} \qquad (3.1)$$

for $t>0$ and $\alpha\in(0,1]$. If f is α-differentiable in some $(0,a)$, $a>0$ and $\lim_{t\to 0^+} D^{\alpha}(f)(t)$ exists, then

$$D^{\alpha}(f)(0) = \lim_{t\to 0^+} D^{\alpha}(f)(t).$$

If f is differentiable, then we have

$$D^{\alpha}(f)(0) = t^{1-\alpha}\frac{d\,f(t)}{dt}\ . \qquad (3.2)$$

Definition 2.3 [Khalil $et.al.$ 2014] (Conformable integral)
Let $a\geq 0$, $t\geq a$, and let f be a function defined on $(a,t]$ where $\alpha\in(0,1]$. Then, the α-differentiable integral of f is defined by,

$$I_a^{\alpha} f(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}}\,dx \qquad (3.3)$$

provided the Riemann improper integral exists.

Definition 3.3 [Anderson 2014]
Let $\alpha\in(0,\infty]$ and $0\leq a\leq b$. A function $f:[\mathrm{a},\mathrm{b}]\to\mathbb{R}$ is α-integrable on $[\mathrm{a},\mathrm{b}]$ if the integral

$$\int_a^b f(t)\,d_{\alpha}t = \int_a^b f(t)\,t^{\alpha-1}\,dt \qquad (3.4)$$

exists.

Definition 3.4 [Andrews $et.al.$ 1999] (Beta function)

$$B(u,v) = \int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}\,dt = \int_0^1 s^{u-1}(1-s)^{v-1}\,ds$$

The well-known Hilbert's inequality and its equivalent form are given as :

Theorem 3.1 [Hardy $et.al.$ 1934]
If $f,g\ \geq 0$, such that

$$0<\int_0^{\infty} f^2(x)\,dx<\infty$$

and

$$0<\int_0^{\infty} g^2(y)\,dy<\infty,$$

then the Hilbert's integral inequality is

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}\,dx\,dy < \pi\left(\int_0^{\infty} f^2(x)\,dx\right)^{1/2}\left(\int_0^{\infty} g^2(y)\,dx\right)^{1/2} \qquad (3.5)$$

and

$$\int_0^{\infty}\left(\int_0^{\infty}\frac{f(x)}{x+y}\,dx\right)^2 dy < \pi^2\int_0^{\infty} f^2(x)\,dx \qquad (3.6)$$

where the constants factor π and π^2 are the best possible. Inequality (3.5) is the well-known Hilbert's inequality.

In 1925, Hilbert's integral inequality has been generalized by Hardy-Riesz as the following result.

Theorem 3.2 [Folland 1999]
If $p>1$, $\frac{1}{p}+\frac{1}{q}\ =1$ and f,g are non-negative real functions such that

$$0<\int_0^{\infty} f^p(x)\,dx<\infty$$

and

$$0 < \int_0^\infty g^q(y)\, dy < \infty ,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\, dx\, dy$$

$$< \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\int_0^\infty f^p(x)\, dx \right)^{\frac{1}{p}}$$

$$X \left(\int_0^\infty g^q(y)\, dy \right)^{1/q} \qquad (3.7)$$

where the constant factor $\pi\, cosec(\pi/\mathrm{p})$ is the best possible.

Inequality (3.1) has been studied and generalized in many directions by a number of distinguished mathematicians.

See Abu Hany (1)2014, Hany (2)2014, Miao. They expended considerable effort in finding best possible constants for the new inequalities they deduced.

4. Main result

The main objective of this paper is obtaining some new extensions of Hilbert's type inequality and some important applications by using some conformable calculus. Where the strict inequality considered and the best possible constant is determined in general and for some other special cases.

Firstly, we shall prove some lemmas which play crucial roles in proving our main results. The following one is useful in establishing our results.

Lemma 4.1

If $q > 1$ and $\alpha \in (0,1]$, we get

$$\int_0^\infty \frac{1}{(x+y)^\alpha} \left(\frac{x}{y}\right)^{\frac{1}{q}} d_\alpha y$$

$$= \frac{\Gamma(\alpha - (1/q))\Gamma(1/q)}{\Gamma(\alpha)} \; ; x > 0 .$$

Proof:

Let $y = \frac{xz}{\alpha}$, $d_\alpha y = \left(\frac{x}{\alpha}\right)^\alpha d_\alpha z \; ; x > 0,$

then we have

$$\int_0^\infty \frac{1}{(x+y)^\alpha} \left(\frac{x}{y}\right)^{1/q} d_\alpha y$$

$$= \int_0^\infty \frac{1}{\left(x + \frac{xz}{\alpha}\right)^\alpha} \left(\frac{\alpha x}{xz}\right)^{1/q} \left(\frac{x}{\alpha}\right)^\alpha d_\alpha z$$

$$\int_0^\infty \frac{1}{\left(\frac{x}{\alpha}\right)^\alpha (\alpha+z)^\alpha} \left(\frac{\alpha}{z}\right)^{1/q} \left(\frac{x}{\alpha}\right)^\alpha d_\alpha z$$

$$= \int_0^\infty \frac{1}{(\alpha+z)^\alpha} \left(\frac{\alpha}{z}\right)^{1/q} d_\alpha z$$

$z = \alpha y,\; d_\alpha z = \alpha^\alpha d_\alpha y$ then we have

$$\int_0^\infty \frac{1}{(\alpha+z)^\alpha} \left(\frac{\alpha}{z}\right)^{\frac{1}{q}} d_\alpha z$$

$$= \int_0^\infty \frac{1}{(\alpha+\alpha y)^\alpha} \left(\frac{\alpha}{\alpha y}\right)^{\frac{1}{q}} \alpha^\alpha d_\alpha y$$

$$= \int_0^\infty \frac{1}{(1+y)^\alpha} \left(\frac{1}{y}\right)^{1/q} d_\alpha y$$

$$= \int_0^\infty \frac{y^{-1/q}}{(1+y)^\alpha} d_\alpha y = \int_0^\infty \frac{y^{-1/q}}{(1+y)^\alpha} y^{\alpha-1} dy$$

$$\int_0^\infty \frac{1}{(x+y)^\alpha} \left(\frac{x}{y}\right)^{\frac{1}{q}} d_\alpha y$$

$$= \int_0^\infty \frac{y^{\alpha - \left(\frac{1}{q}\right) - 1}}{(1+y)^{\left(\alpha - \left(\frac{1}{q}\right)\right) + \left(\frac{1}{q}\right)}} dy$$

$$= \beta(\alpha - (1/q), 1/q\,)$$

$$= \frac{\Gamma(\alpha - (1/q))\Gamma(1/q)}{\Gamma(\alpha)} .$$

In the following, we present a new fractional form to the well-known Hilbert's inequality (3.5).

Theorem 4.2

If f, g are non-negative real functions such that

$$0 < \int_0^\infty f^2(x) d_\alpha x < \infty,$$

$0 < \int_0^\infty g^2(y) d_\alpha y < \infty$

and for $p > 1, \frac{1}{p} + \frac{1}{q} = 1,\ \alpha \in (0,1]$, then we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\alpha} d_\alpha x d_\alpha y < \frac{\Gamma\left(\alpha - \left(\frac{1}{2}\right)\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha)} \left(\int_0^\infty f^2(x)\, d_\alpha x\right)^{\frac{1}{2}} X \left(\int_0^\infty g^2(y)\, d_\alpha y\right)^{1/2} \quad (4.1)$$

where the constant factor $\frac{\Gamma(\alpha-(1/2))\Gamma(1/2)}{\Gamma(\alpha)}$ is the best possible.

Proof:

For $\alpha \in (0,1]$, we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\alpha} d_\alpha x\, d_\alpha \le \int_0^\infty \int_0^\infty \frac{f(x)}{(x+y)^{\frac{\alpha}{2}} \left(\frac{y}{x}\right)^{\frac{1}{4}}} \frac{g(y)}{(x+y)^{\frac{\alpha}{2}} \left(\frac{x}{y}\right)^{\frac{1}{4}}} d_\alpha x\, d_\alpha y.$$

.

By Cauchy Schwarz inequality for integrals,

$$\le \left[\int_0^\infty \int_0^\infty \frac{f^2(x)}{(x+y)^\alpha \left(\frac{x}{y}\right)^{\frac{1}{2}}} d_\alpha x d_\alpha y\right]^{\frac{1}{2}} X \left[\int_0^\infty \int_0^\infty \frac{g^2(y)}{(x+y)^\alpha \left(\frac{x}{y}\right)^{\frac{1}{2}}} d_\alpha x d_\alpha y\right]^{\frac{1}{2}}$$

$$\le \left[\int_0^\infty f^2(x) \int_0^\infty \frac{1}{(x+y)^\alpha \left(\frac{y}{x}\right)^{\frac{1}{2}}} d_\alpha y d_\alpha x\right]^{\frac{1}{2}} X \left[\int_0^\infty g^2(y) \int_0^\infty \frac{1}{(x+y)^\alpha \left(\frac{x}{y}\right)^{1/2}} d_\alpha x d_\alpha y\right]^{1/2}.$$

From Lemma 4.1, and let $p = q = 2$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\alpha} d_\alpha x d_\alpha y$$

$$\le \left[\frac{\Gamma\left(\alpha - \left(\frac{1}{2}\right)\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha)} \int_0^\infty f^2(x)\, dx\right]^{\frac{1}{2}} X \left[\frac{\Gamma(\alpha-(1/2))\Gamma(1/2)}{\Gamma(\alpha)} \int_0^\infty g^2(y)\, dy\right]^{1/2}$$

$$= \left|\frac{\Gamma\left(\alpha - \left(\frac{1}{2}\right)\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha)}\right|^{\frac{1}{2}} \left|\frac{\Gamma\left(\alpha - \left(\frac{1}{2}\right)\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha)}\right|^{\frac{1}{2}} X \left(\int_0^\infty f^2(x)\, dx\right)^{1/2} \left(\int_0^\infty g^2(y)\, dy\right)^{1/2}$$

$$= \frac{\Gamma\left(\alpha - \left(\frac{1}{2}\right)\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha)} \left(\int_0^\infty f^2(x)\, dx\right)^{\frac{1}{2}} X \left(\int_0^\infty g^2(y)\, dy\right)^{1/2}.$$

The following theorem is a generalization of Theorem 4.2 .

Theorem 4.3

If f and g are non-negative real functions and $f, g \in L_2(0,\infty)$ such that

$$0 < \int_0^\infty f^p(x)\, d_\alpha x < \infty,$$

$0 < \int_0^\infty g^q(y)\, d_\alpha y < \infty,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,$ and $\alpha \in (0,1]$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\alpha} d_\alpha x d_\alpha y < A\left(\int_0^\infty f^p(x)\, d_\alpha x\right)^{1/p} \left(\int_0^\infty g^q(y)\, d_\alpha y\right)^{1/q} \quad (4.2)$$

where A is the best constant factor, such that

$$A = \frac{1}{\Gamma(\alpha)}\left|\Gamma\left(\alpha - \left(\frac{1}{q}\right)\right)\Gamma\left(\frac{1}{q}\right)\right|^{\frac{1}{p}}$$
$$X \ |\Gamma(\alpha - (1/p))\Gamma(1/p)|^{1/q} \ .$$

Proof:

For $\alpha \in (0,1]$, we get

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\alpha}\, d_\alpha x\, d_\alpha y$$
$$\leq \int_0^\infty\int_0^\infty \frac{f(x)}{(x+y)^{\frac{\alpha}{p}}\left(\frac{y}{x}\right)^{\frac{1}{pq}}}\frac{g(y)}{(x+y)^{\frac{\alpha}{q}}\left(\frac{x}{y}\right)^{\frac{1}{pq}}}\, d_\alpha x\, d_\alpha y \ .$$

By Hölder's inequality for integrals,

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\alpha}\, d_\alpha x d_\alpha y$$
$$\leq \left[\int_0^\infty\int_0^\infty \frac{f^p(x)}{(x+y)^\alpha\left(\frac{y}{x}\right)^{\frac{1}{q}}}\, d_\alpha x d_\alpha y\right]^{\frac{1}{p}}$$
$$X \left[\int_0^\infty\int_0^\infty \frac{g^q(y)}{(x+y)^\alpha\left(\frac{x}{y}\right)^{1/p}}\, d_\alpha x d_\alpha y\right]^{1/q}$$
$$= \left[\int_0^\infty f^p(x)\int_0^\infty \frac{1}{(x+y)^\alpha\left(\frac{y}{x}\right)^{\frac{1}{q}}}\, d_\alpha y d_\alpha x\right]^{\frac{1}{p}}$$
$$X \left[\int_0^\infty g^q(y)\int_0^\infty \frac{1}{(x+y)^\alpha\left(\frac{x}{y}\right)^{1/p}}\, d_\alpha x d_\alpha y\right]^{1/q}$$
$$= \left[\frac{\Gamma\left(\alpha - \left(\frac{1}{q}\right)\right)\Gamma\left(\frac{1}{q}\right)}{\Gamma(\alpha)}\int_0^\infty f^p(x)\, d_\alpha x\right]^{\frac{1}{p}}$$
$$X \left[\frac{\Gamma(\alpha - (1/p))\Gamma(1/p)}{\Gamma(\alpha)}\int_0^\infty g^q(y)\, d_\alpha y\right]^{1/q}$$
$$= \left|\frac{\Gamma\left(\alpha - \left(\frac{1}{q}\right)\right)\Gamma\left(\frac{1}{q}\right)}{\Gamma(\alpha)}\right|^{\frac{1}{p}}\left|\frac{\Gamma\left(\alpha - \left(\frac{1}{p}\right)\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma(\alpha)}\right|^{\frac{1}{q}}$$
$$X \left(\int_0^\infty f^p(x)\, d_\alpha x\right)^{1/p}\left(\int_0^\infty g^q(y)\, d_\alpha y\right)^{1/q} .$$

Thus, we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\alpha}\, d_\alpha x d_\alpha y$$
$$\leq A\left(\int_0^\infty f^p(x)\, d_\alpha x\right)^{1/p}\left(\int_0^\infty g^q(y)\, d_\alpha y\right)^{1/q} . \quad (4.3)$$

Remark : The equality in inequality (4.3) can be achieved in two possibilities. The first possibility, if f or g is null that would contradict with one of the hypotheses. The second possibility, if both are effectively proportional implies that for almost all x and y there exist constant c and d they are not all zero, without lose the generality, suppose $c \neq 0$, and

$$c\left(\left|\frac{f(x)}{(x+y)^{\frac{1}{p}}}\right|\left(\frac{x}{y}\right)^{\frac{1}{pq}}\right)^{\frac{1}{q}}$$
$$= d\left(\left|\frac{g(y)}{(x+y)^{1/q}}\right|\left(\frac{y}{x}\right)^{1/pq}\right)^{1/p}$$

We conclude that

$$c|f(x)|^{1/q}\left(\frac{x}{y}\right)^{1/pq^2} = d|g(y)|^{1/p}\left(\frac{y}{x}\right)^{1/p^2q}$$
$$Cf^p(x)\left(\frac{x}{y}\right)^{1/q} = Dg^q(y)\left(\frac{y}{x}\right)^{1/p}$$
$$= constant$$
$$C\, x\, f^p(x) = D\, y\, g^q(y) = constant$$

where $C = c^{pq}$ and $D = d^{pq}$.

Consequently,

$$C\, x\, f^p(x) = constant$$
$$f^p(x) = b\, x^{-1}$$

where $b = constant/C$, thus

$$\int_0^\infty f^p(x)\, d_\alpha x = b\int_0^\infty x^{-1}\, d_\alpha x$$
$$= b\int_0^\infty x^{-1}x^{\alpha-1}\, dx$$

$$\int_0^\infty f^p(x)\, d_\alpha x = b\int_0^\infty x^{\alpha-2}\, dx$$
$$= b\int_0^\infty \frac{1}{x^{2-\alpha}}\, dx .$$

Since $\int_0^\infty \frac{1}{x^{2-\alpha}}dx$ diverges which contradicts the fact that $f \in L_2(0,\infty)$.
Hence inequality (4.3) takes the form of strict inequality, so we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y} d_\alpha x d_\alpha y < A\left(\int_0^\infty f^p(x)\, d_\alpha x\right)^{1/p}\left(\int_0^\infty g^q(y)\, d_\alpha y\right)^{1/q} .$$

Lemma 4.4

If $p > 1$ and $\alpha \in (0,1]$, then

$$\int_0^\infty \frac{1}{(z+\alpha)^\alpha}\left(\frac{\alpha}{z}\right)^{\alpha/p} d_\alpha z = \frac{\Gamma(\alpha-(\alpha/p))\Gamma(\alpha/p)}{\Gamma(\alpha)}$$

Proof:

Setting $z = \frac{\alpha}{y}$, then $d_\alpha z = -\frac{\alpha^\alpha}{y^{2\alpha}} d_\alpha y$. Therefore

$$\int_0^\infty \frac{1}{(z+\alpha)^\alpha}\left(\frac{\alpha}{z}\right)^{\frac{\alpha}{p}} d_\alpha z$$
$$= \int_\infty^0 \frac{1}{\left(\frac{\alpha}{y}+\alpha\right)^\alpha}\left(\frac{\alpha y}{\alpha}\right)^{\frac{\alpha}{p}}\left(-\frac{\alpha^\alpha}{y^{2\alpha}}\right) d_\alpha y$$
$$= \int_0^\infty \frac{1}{(y+1)^\alpha} y^{(\alpha/p)-\alpha} d_\alpha y$$
$$= \int_0^\infty \frac{y^{\left(\frac{\alpha}{p}\right)-\alpha}}{(y+1)^\alpha} d_\alpha y = \int_0^\infty \frac{y^{\left(\frac{\alpha}{p}\right)-\alpha}\, y^{\alpha-1}}{(y+1)^\alpha} dy$$
$$= \int_0^\infty \frac{y^{(\alpha/p)-1}}{(y+1)^{\alpha-(\alpha/p)+(\alpha/p)}} dy .$$

From the definition of Beta function,

$$\int_0^\infty \frac{1}{(z+\alpha)^\alpha}\left(\frac{\alpha}{z}\right)^{\alpha/p} d_\alpha z$$
$$= B(\alpha-(\alpha/p),(\alpha/p))$$
$$= \frac{\Gamma(\alpha-(\alpha/p))\Gamma(\alpha/p)}{\Gamma(\alpha)} .$$

The following theorem presents an equivalent form to inequality (3.6).

Theorem 4.5

If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\alpha \in (0,1]$ and f, g are non-negative real functions such that $0 < \int_0^\infty f^p(x)d_\alpha x < \infty$ and $0 < \int_0^\infty g^q(y)d_\alpha y < \infty$, then

$$\int_0^\infty\left(\int_0^\infty \frac{f(x)}{(x+y)^\alpha} d_\alpha x\right)^p d_\alpha y < C^p\int_0^\infty f^p(x)\, d_\alpha x \qquad (4.4)$$

$$\int_0^\infty\left(\int_0^\infty \frac{g(y)}{(x+y)^\alpha} d_\alpha y\right)^q d_\alpha x < C^q\int_0^\infty g^q(y)\, d_\alpha y \qquad (4.5)$$

where the constant factors C^p and C^q are the best possible, and
$C = \frac{\Gamma(\alpha-(\alpha/p))\Gamma(\alpha/p)}{\Gamma(\alpha)}$.

Proof : Let

$$g(y) = \left(\int_0^\infty \frac{f(x)}{(x+y)^\alpha} d_\alpha x\right)^{\frac{p}{q}} .$$

Now

$$\int_0^\infty\left(\int_0^\infty \frac{f(x)}{(x+y)^\alpha} d_\alpha x\right)^p d_\alpha y$$
$$= \int_0^\infty\left(\int_0^\infty \frac{f(x)}{(x+y)^\alpha} d_\alpha x\right)^{\frac{p}{q}}\left(\int_0^\infty \frac{f(x)}{(x+y)^\alpha} d_\alpha x\right) d_\alpha y$$
$$= \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\alpha} d_\alpha x\, d_\alpha y .$$

By using Theorem 4.3 , equation 4.2, we have

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{(x+y)^\alpha} d_\alpha x \right)^p d_\alpha y$$

$$\leq A \left(\int_0^\infty f^p(x)\, d_\alpha x \right)^{1/p} \left(\int_0^\infty g^q(y)\, d_\alpha y \right)^{1/q}$$

$$= A \left(\int_0^\infty f^p(x)\, d_\alpha x \right)^{\frac{1}{p}} \left(\int_0^\infty \left(\int_0^\infty \frac{f(x)}{(x+y)^\alpha} d_\alpha x \right)^p d_\alpha y \right)^{\frac{1}{q}},$$

then

$$\left(\int_0^\infty \left(\int_0^\infty \frac{f(x)}{(x+y)^\alpha} d_\alpha x \right)^p d_\alpha y \right)^{1-\frac{1}{q}}$$

$$\leq A \left(\int_0^\infty f^p(x)\, d_\alpha x \right)^{\frac{1}{p}}.$$

Hence

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{(x+y)^\alpha} d_\alpha x \right)^p d_\alpha y$$

$$\leq A^p \int_0^\infty f^p(x)\, d_\alpha x.$$

We claim that for $\alpha = 1$, we obtain the results that has been proved and published before in this field .

References:

Abu Hany A. A., (2014): "On Some Integral Inequalities Analogues to Hilbert's Inequality". *Gen. Math. Notes, 20(1)*, pp.58-66.

Abu Hany A. A., (2014): "On Some New Analogues of Hilbert's Inequality". International Journal of Mathematics and Computation, 24(3), pp.70-76.

Anderson R. D., (2014): " Taylor's Formula and Integral Inequalities for Conformable Fractional Derivatives" arXiv: 1409.5888 vl.

Folland, G. B., (1999): " Real Analysis. Modern Techniques and Their Applications", John Wiley and Sons, New York, United States of American.

Hardy, G. H., Littlewood, J. E. and Polya, G., (1934): " Inequalities", Cambridge University Press, Cambridge, United Kingdom.

Khalil R., Al Horani M., Yousef A. and Sababheh M., (2014): "A new definition of fractional derivative", *Journal of Computational and Applied Mathematics*, pp. 65–70.

Li, Y., Qian, Y., and He, B., (2007) "On Further Analogs of Hilbert's Inequality". International Journal of Mathematics and Mathematical Sciences, Volume 2007, pp.1-6.

Miao Y. and Du H.X., (2009): "Several analogues of Hilbert inequalities", *Demonstratio Math, XLII (2),* pp. 297-302.

Miao Y. and Du H.X., (2010): "A note on Hilbert type integral inequality", Inequality Theory and Applications, 6(2010), pp. 261-267.

Andrews G.E., Askey R. and Roy R.,(1999): "Special Functions". Encyclopedia of Mathematics and its Applications", vol. 71, *Cambridge University Press*.