Reliability Function of the Connected- (2,2)-out-of-(*m*,*n*): F Linear and Circular System

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ملخص :

يتكون النظام التتابعي connected-(2,2)-out-of-(m,n):F الشبكي الدائري والخطي من عدد $m \times n$ جزء، النظام يفشل في حالة فشل مربع متصل مكون من 2×2 جزء. في هذا البحث تم ايجاد خوارزمية جديدة بطريقة استرجاع الرتبة الاقل لحساب اقتران موثوقية (امكانية عمل) هذا النظام التتابعي الخطي والدائري الشبكي، هذه الخوارزمية تعتمد على ايجاد حالات نجاح وفشل النظام التتابعي 2-out-of-n: F الخطي والدائري احادي البعد.

Abstract

The connected-(2,2)-out-of-(m,n):F linear (circular) lattice system consists of $m \times n$ *components. The system fails if and only if an area of connected failed components includes a square* 2×2 *of connected failed components. In this paper a new recursive algorithm to find the reliability function of the connected- (2,2)-out-of-(m,n): F linear and circular system is obtained. The algorithm depends on the functioning and the failure state of the consecutive 2-out-of-n: F linear and circular system.*

Keywords: The consecutive *k*-out-of-*n*: F system, The connected (*r*,*s*)-out-of-(*m*,*n*): F system.

Notation:

L(C)	: Linear (Circular)
$\mathbb{I}_n^1$	: $=\{1,2,...,n\}$
$P\left(\mathbb{I}_n^1\right)$	: The power set of $\mathbb{I}_n^1$.
$\Theta_{L(C)}$	: The set of all functioning sets of the consecutive 2-out-of-*n*: F linear (circular) system
$\Psi_{L(C)}$	: The set of all failed sets of the consecutive 2-out-of-*n*: F linear (circular) system
$p_j\left(q_j\right)$	: The reliability (unreliability) of the j^{th} component, and $p_W=\prod_{j\notin W}p_j, q_W=\prod_{j\in W}q_j \;\; \forall W\subseteq \mathbb{I}_n^1$
$p_{ij}\left(q_{ij}\right)$	: The reliability (unreliability) of the j^{th} component at the i^{th} layer (circle).
$p_W^i\left(q_W^i\right)$	: The reliability (unreliability) of the i^{th} layer (circle), when *W* represents the i^{th} layer (circle) $p_W^i=\prod_{j\notin W}p_{ij}, q_W^i=\prod_{j\in W}q_{ij} \; : W\subseteq \mathbb{I}_n^1$
p_n^s	: A function of parameters *p*, *n*, *s*, where $p_n^s=p(n,s)=p^{n-s}q^s$
$\mathbb{R}(W)$	: Probability (reliability) the consecutive 2-out-of-*n*: F linear (circular) system represented by a set *W*. $R(W)=p_W q_W$
$\mathbb{R}_{L(C)}(\Theta)$	: The reliability of the consecutive 2-out-of-*n*: F linear (circular) system, $=\sum_{W\in\Theta_{L(C)}}R(W)$
$\mathbb{R}_{L(C)}(i)$	: The reliability function of the connected-(2,2)-out-of-(*i*,*n*): F linear (circular) subsystem
$\mathbb{R}_{L(C)}(i,X)$	: The reliability function of the connected-(2,2)-out-of-(*i*,*n*): F linear (circular) subsystem when *X* represents the i^{th} layer (circle).
$\mathbb{R}_{L(C)}(i,\Theta)$	: The reliability function of the connected-(2,2)-out-of-(*i*,*n*): F linear (circular) subsystem, when any $X\in\Theta_{L(C)}$ represents the i^{th} layer (circle).
$\mathbb{R}_{L(C)}(i,\Psi)$	: The reliability function of the connected-(2,2)-out-of-(*i*,*n*): F linear (circular) subsystem, when any $X\in\Psi_{L(C)}$ represents the i^{th} layer (circle).

1. Introduction

The consecutive *k*-out-of-*n*: F system is specified by the number of connected components, *n*, and the minimum number of consecutive failed components, $k \le n$, that transform the system from the functioning to the failure state. The system was firstly studied by Kontoleon [1], and then by Chiang and Niu [2]. It is classified according to the connection between components into two types: linear and circular. Many researchers have extensively studied the reliability of the system, and many generalizations were appeared (Derman et al. [3], Malinowski and Preuss [4], and Yamamoto and Akiba [5].

The connected-(*r*,*s*)-out-of-(*m*,*n*):F linear & circular system consists of $m \times n$ components, the components of the linear system arranged as the elements of a (*m*,*n*)-matrix, while the circular system consists of *m* circles and *n* rays, (the circles have the same center, the intersections of circles and rays represent the elements, where 'ray' means a line from the center of the circles to their perimeters) , see figure 3.1. The system fails if and only if all components in a connected (*r*,*s*)-submatrix fail. This system is considered as a type of 2-dimnetional version of the consecutive *k*-out-of-*n*: F linear (circular) systems, which was firstly introduced by Salvia & Lasher [6], and then generalized by Boehme et al. [7].

Studying the reliability of the 2 dimensional consecutive system is very complicated compared with the one dimensional consecutive system. Yamamoto and Miyakawa [8] and [9] found the reliability of the linear and the circular system using recursive algorithms respectively, whereas Yamamoto and Akiba [10] introduced a recursive algorithm to find the reliability of the circular system only, they considered the system as a cylindrical grid and made a vertical cut between the 1^{st} and the n^{th} rays, and then treated the circular system as a linear system. It is worth mentioning that the connected (*r*,*s*)-out of-(*m*,*n*): F system has diverse applications such as the design of electronic devices, disease diagnosis on the *X*-ray, pattern detection [9], 'Feelers for measuring temperature on reaction chamber' [7].

Nashwan [13] obtained an algorithm to find the failure function of the connected (2,2)-out-of-(*m*,*n*): F linear and circular lattice system, while this paper use the same technique to obtain the complement, "the reliability function of the connected (2,2)-out-of-(*m*,*n*): F linear and circular lattice system using recursive algorithm".

The following assumptions are assumed to be satisfied by the connected (2,2)-out-of-(*m*,*n*): F linear and circular lattice system.

1. The state of the component and the system is either "functioning" or "failed"
2. All the components are mutually statistically independent.

2. The consecutive 2-out-of-*n*: F system

In this section, the Index Structure Function [11-13] is used to present the consecutive 2-out-of-*n*: F system. Let $\mathbb{I}_n^1$ denotes the possible labels (indices) of the components of the consecutive 2-out-of-*n*: F linear and circular system, hence the failure space of the components is $P\left(\mathbb{I}_n^1\right)$. The set $X \in P\left(\mathbb{I}_n^1\right)$ represents the system, and consists of all indices of failed components, is called "functioning set" if no two consecutive indices of failed components are included in *X*, otherwise, *X* is called "failed set", (e.g. in the consecutive 2-out-of 6: F linear and circular system, the set $X = \{1,2,5\} \subset \mathbb{I}_6^1$ or for simply *X*=125, indicates that the 1st, the 2nd and the 5th components are in the failure state, hence *X* is a failed set, but the set *X*=135 is a functioning set. On the whole, we can partition $P\left(\mathbb{I}_n^1\right)$ into two disjoint sub collections, $\Theta_{L(C)}$ and $\Psi_{L(C)}$ the functioning and failure space of the linear (circular) system respectively, where $\Theta_{L(C)}$ is defined as the set of all functioning sets of the linear (circular) system, and $\Psi_{L(C)} = \left\{X \in P\left(\mathbb{I}_n^1\right) : \{i, i+1\} \subseteq X\right\}$ where $i \in \mathbb{I}_{n-1}^1$ (for the circular system $i \in \mathbb{I}_n^1$ where the (*n*+1)th component is the 1st component, and the (*n*+2)th component is the 2nd component …, etc.).

3. The Proposed Algorithm

Consider the connected-(2,2)-out-of-(*m*,*n*): F linear (circle) system , the system fails if a square of 2×2 connected components are failed, if $\mathbb{I}_n^1$ denotes the label of the components for any layer (circle), hence $P\left(\mathbb{I}_n^1\right) = \Theta_{L(C)} \cup \Psi_{L(C)}$ is the failure space of the components for any layer (circle), if the event "$X, Y \in P\left(\mathbb{I}_n^1\right)$ represent any two consecutive layers (circles), and $X \cap Y \in \Psi_{L(C)}$" occurs, then the system fails. (see figure 3.1).

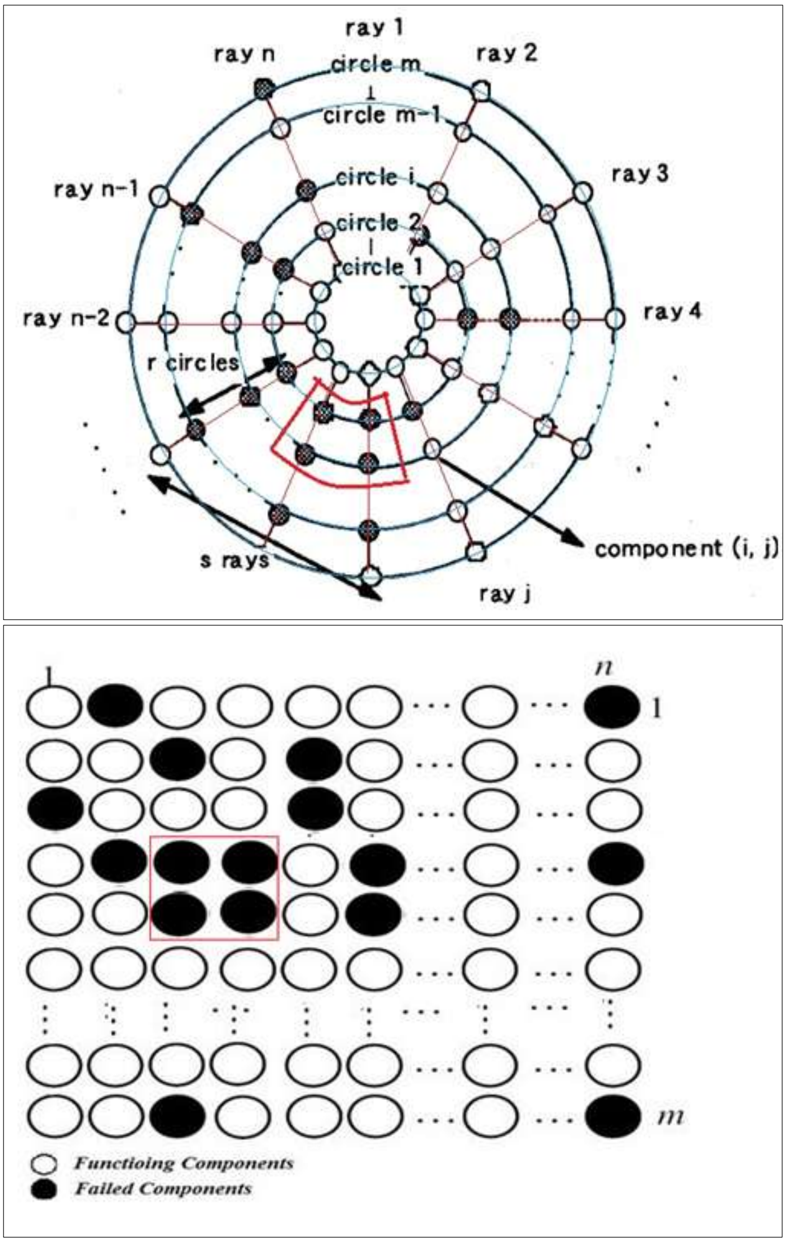

Figure 3.1: Example of a failed connected-(2,2)-out-of-(*m*,*n*):F linear and circular systems.

Now, assume the connected-(2,2)-out-of-(i,n): F linear (circle) subsystem is in the functioning state, add the (i+1)$^{\text{th}}$ layer (circle) to the (i, n) subsystem, keeping the resulted (i+1,n) subsystem is also in the functioning state. If $X,Y\in P\left(\mathbb{I}_n^1\right)$ represent the (i+1)$^{\text{th}}$ and i^{th} layers (circles) respectively, this implies that the area of the connected failures in the consecutive layers (circle) is not including a connected square of 2×2 failed components, i.e. $X\bigcap Y\notin\Psi_{L(C)}$ (see figure 3.3 and 3.2.*a* ,*b*,*c*). In general, we have the following cases:

1. If $X\in\Theta_{L(C)}$, then for all $Y\in P\left(\mathbb{I}_n^1\right)$, we have $X\bigcap Y\notin\Psi_{L(C)}$, which implies that X is not a cause for failure i.e. (i+1,n) subsystem is in the functioning state. (see figure 3.2.*a*)
2. If $X\in\Psi_{L(C)}$, then for all $Y\in\Theta_{L(C)}$, we have $X\bigcap Y\notin\Psi_{L(C)}$, which implies that X is not a cause for failure i.e. the (i+1,n) subsystem is in the functioning state. (see figure 3.2.*b*)
3. If $X\in\Psi_{L(C)}$, the (i+1, n) subsystem is in the functioning state for some $Y\in\Psi_{L(C)}$, such that $X\bigcap Y\notin\Psi_{L(C)}$, otherwise $X\bigcap Y\in\Psi_{L(C)}$ which implies that the (i+1,n) subsystem fails, which contradicts our assumption (see figure 3.2.*c*). To restrict all these cases, define $\Theta_{L(C)}\left(X\right)=\left\{Y\in P\left(\mathbb{I}_n^1\right):X\bigcap Y\notin\Psi_{L(C)}\right\}$ and $\Psi_{L(C)}\left(X\right)=\left\{Y\in\Psi_{L(C)}:X\bigcap Y\notin\Psi_{L(C)}\right\}$. Note that, anyone can easily check the following:

1. $\Theta_{L(C)}\left(X\right)=\Theta_{L(C)}\bigcup\Psi_{L(C)}\left(X\right): \qquad \forall X\in\Psi_{L(C)}$.
2. $\Theta_{L(C)}\left(X\right)=P\left(\mathbb{I}_n^1\right):\forall X\in\Theta_{L(C)}$
3. $\Theta_{L(C)}\left(\mathbb{I}_n^1\right)=\Theta_{L(C)},\Psi_{L(C)}\left(\mathbb{I}_n^1\right)=\{\ \}$.
4. $\Theta_{L(C)}\subseteq\Theta_{L(C)}\left(X\right):\forall X\in P\left(\mathbb{I}_n^1\right)$

Now, for i=1,2,…, m-1, define the random variable Z_{i+1} on $P\left(\mathbb{I}_n^1\right)$ such that $\left\{Z_{i+1}=X\right\}$ the event "the (i+1)$^{\text{th}}$ layer (circle) is represented by $X\in P\left(\mathbb{I}_n^1\right)$", and $A_{i+1}\left(X\right)$ the event "the connected-(2,2)-out-of-

(i+1,n): F linear (circle) subsystem is in the functioning state, where the set X represents the $(i+1)^{th}$ layer (circle)". Note that $\forall X \in P(\mathbb{I}_n^1)$

$$A_{i+1}(X)=\begin{cases}\{Z_{i+1}=X\} & : i=0\\ \{Z_{i+1}=X\}\bigcap\left[\bigcup_{Y\in\Theta_{L(C)}(X)} A_i(Y)\right] & : i=1,2,3,\dots,m-1\end{cases}$$

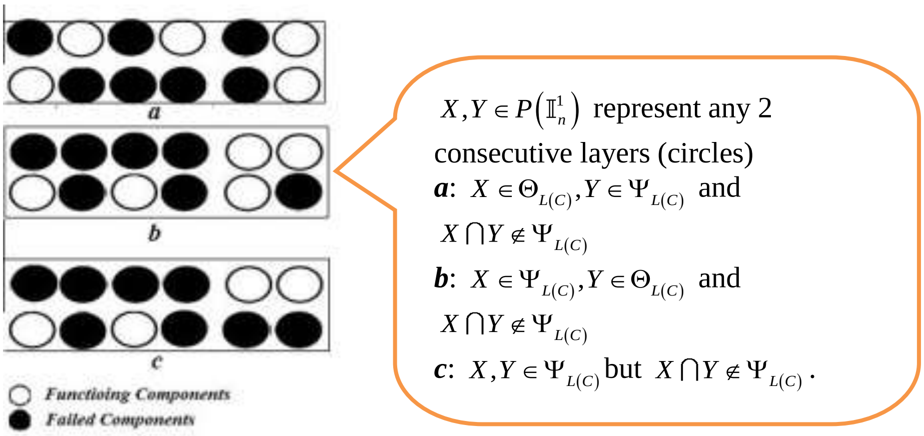

Figure 3.2: Example of a functioning state of the layers (circles) in the connected-(2,2)-out-of-(m,n):F systems.

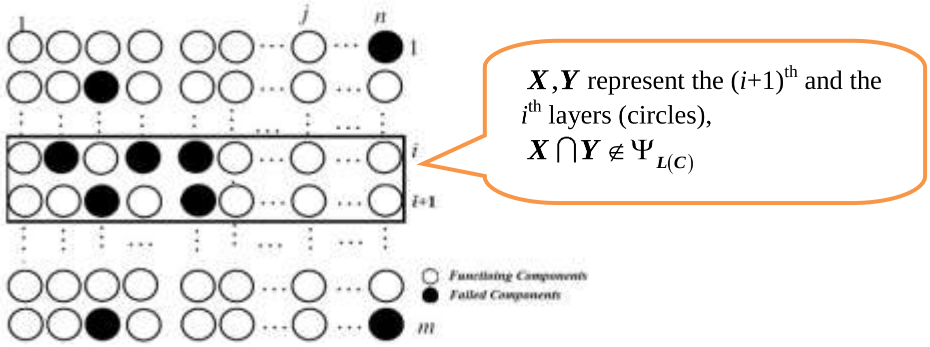

Figure 3.3: Example of the functioning connected-(2,2)-out-of-(m,n):F system.

Theorem 3.1: Consider the connected (2,2)-out-of-(m,n): F linear (circular) system, in the functioning state, then

1. For all $X\in P(\mathbb{I}_n^1)\Rightarrow\mathbb{R}_{L(C)}(1,X)=\mathbb{R}(X)$.
2. For i=1,2,…, m-1, then the reliability of the (i+1,n) subsystem,

$$\mathbb{R}_{L(C)}(i+1,\Theta)=\mathbb{R}_{L(C)}(i)\mathbb{R}_{L(C)}(\Theta)$$

3. For i=1,2,…, m-1, and for all $X\in\Psi_{L(C)}$

$$\mathbb{R}_{L(C)}(i+1,X)=\mathbb{R}(X)\left[\mathbb{R}_{L(C)}(i,\Theta)+\sum_{Y\in\Psi_{L(C)}(X)}\mathbb{R}(i,Y)\right]$$ and

$$\mathbb{R}_{L(C)}(i+1,\Psi)=\sum_{X\in\Psi_{L(C)}}\mathbb{R}_{L(C)}(i+1,X)$$

4. For i=1,2,…,m, then

$$\mathbb{R}_{L(C)}(i)=\mathbb{R}_{L(C)}(i,\Psi)+\mathbb{R}_{L(C)}(i,\Theta)$$

Proof:

1. For all $X\in P\left(\mathbb{I}_n^1\right)$, if i=1, then the system consists of one layer (circular), i.e. there is no failure, i.e. $\mathbb{R}_{L(C)}(1,X)=P\{A_1(X)\}=P\{Z_1=X\}=\mathbb{R}_{L(C)}(X)$.
2. For i=1,2, 3,…,m-1, and for any $X\in\Theta_{L(C)}$,

$$\mathbb{R}_{L(C)}(i+1,X)=P\{A_{i+1}(X)\}$$

$$=P\left\{\{Z_{i+1}=X\}\cap\left\{\bigcup_{Y\in\Theta_{L(C)}(X)}A_i(Y)\right\}\right\}=P\left\{\bigcup_{Y\in P\left(\mathbb{I}_n^1\right)}\left\{\{Z_{i+1}=X\}\cap\{A_i(Y)\}\right\}\right\}$$

$$=\sum_{Y\in P\left(\mathbb{I}_n^1\right)}P\left\{\{Z_i=X\}\cap\{A_i(Y)\}\right\}=P\{Z_i=X\}\left(\sum_{Y\in P\left(\mathbb{I}_n^1\right)}P\{A_i(Y)\}\right)$$

$$=\mathbb{R}(X)\sum_{Y\in P\left(\mathbb{I}_n^1\right)}\mathbb{R}_{L(C)}(i,Y)=R(X)\mathbb{R}_{L(C)}(i)$$

$$\mathbb{R}_{L(C)}(i+1,\Theta)=P\left\{\bigcup_{X\in\Theta_{L(C)}}A_{i+1}(X)\right\}=\sum_{X\in\Theta_{L(C)}}P\{A_{i+1}(X)\}=\sum_{X\in\Theta_{L(C)}}\mathbb{R}_{L(C)}(i+1,X)$$

$$=\mathbb{R}_{L(C)}(i)\sum_{X\in\Theta_{L(C)}}\mathbb{R}(X)=\mathbb{R}_{L(C)}(i)\mathbb{R}_{L(C)}(\Theta)$$

3. For any $X\in\Psi_{L(C)}$, and for i=1,2, 3,…,m-1,

$$\mathbb{R}_{L(C)}(i+1,X) = P\{A_{i+1}(X)\} = P\left\{\{Z_{i+1}=X\} \cap \left\{\bigcup_{Y\in\Theta_{L(C)}(X)} A_i(Y)\right\}\right\}$$

$$= P\left\{\bigcup_{Y\in\Theta_{L(C)}(X)} \{\{Z_{i+1}=X\}\cap\{A_i(Y)\}\}\right\}$$

$$= \sum_{Y\in\Theta_{L(C)}(X)} P\{\{Z_i=X\}\cap\{A_i(Y)\}\}$$

$$\mathbb{R}_{L(C)}(i+1,X) = P\{Z_i=X\}\left(\sum_{Y\in\Theta_{L(C)}(X)} P\{A_i(Y)\}\right)$$

$$= \mathbb{R}(X)\sum_{Y\in\Theta_{L(C)}(X)} \mathbb{R}_{L(C)}(i,Y)$$

$$= \mathbb{R}(X)\left[\sum_{Y\in\Theta_{L(C)}} \mathbb{R}_{L(C)}(i,Y) + \sum_{Y\in\Psi_{L(C)}(X)} \mathbb{R}_{L(C)}(i,Y)\right]$$

Then $\mathbb{R}_{L(C)}(i+1,X) = \mathbb{R}(X)\left[\mathbb{R}_{L(C)}(i,\Theta) + \sum_{Y\in\Psi_{L(C)}(X)} \mathbb{R}_{L(C)}(i,Y)\right]$ and

$$\mathbb{R}_{L(C)}(i+1,\Psi) = P\left\{\bigcup_{X\in\Psi_{L(C)}} A_{i+1}(X)\right\} = \sum_{X\in\Psi_{L(C)}} P\{A_{i+1}(X)\}$$

$$= \sum_{X\in\Psi_{L(C)}} \mathbb{R}_{L(C)}(i+1,X)$$

4. Finally, it's clear that, for i=1,2,…,m

$$\mathbb{R}_{L(C)}(i) = P\left\{\bigcup_{X\in P(\mathbb{I}_n^1)} A_i(X)\right\} = \sum_{X\in P(\mathbb{I}_n^1)} P\{A_i(X)\} = \sum_{X\in P(\mathbb{I}_n^1)} \mathbb{R}_{L(C)}(i,X) =$$

$$= \sum_{X\in\Psi_{L(C)}} \mathbb{R}_{L(C)}(i,X) + \sum_{X\in\Theta_{L(C)}} \mathbb{R}_{L(C)}(i,X) = \mathbb{R}_{L(C)}(i,\Psi) + \mathbb{R}_{L(C)}(i,\Theta)$$

Now, we propose the algorithm of calculating the reliability of a connected-(2,2)-out-of-(m,n): F linear and circular systems. This algorithm is based on the applications of the results given by Theorem 3.1. illustrative examples will be given. The proposed algorithm consists of the following steps:

1. Consider the consecutive 2-out-of-*n*: F linear and circular system, and set the $\Theta_{L(C)}$ and $\Psi_{L(C)}$
2. For all $X \in \Psi_{L(C)}$, find $\Psi_{L(C)}(X)$.
3. Find $\mathbb{R}_{L(C)}(1,\Theta) = \sum_{Y \in \Theta_{L(C)}} \mathbb{R}_{L(C)}(Y)$,

   and $\mathbb{R}_{L(C)}(1,\Psi) = \sum_{Y \in \Psi_{L(C)}} \mathbb{R}_{L(C)}(Y)$
4. For all $i = 1,2,...,m-1$, $\mathbb{R}_{L(C)}(i+1,\Theta) = \mathbb{R}_{L(C)}(i)\mathbb{R}_{L(C)}(1,\Theta)$.
5. For all $X \in \Psi_{L(C)}$ and $i = 1,2,...,m-1$, find $\mathbb{R}_{L(C)}(i+1,X) = \mathbb{R}(X)\left[\mathbb{R}_{L(C)}(i,\Theta) + \sum_{Y \in \Psi_{L(C)}(X)} \mathbb{R}(i,Y)\right]$ and then, $\mathbb{R}_{L(C)}(i+1,\Psi) = \sum_{X \in \Psi_{L(C)}} \mathbb{R}_{L(C)}(i+1,X)$.
6. For all $i = 1,2,...,m$, $\mathbb{R}_{L(C)}(i) = \mathbb{R}_{L(C)}(i,\Psi) + \mathbb{R}_{L(C)}(i,\Theta)$

Example 3.1: Compute the reliability of connected (2,2)-out-of-(3,4): F circular system, when the components are independent and identically distributed.

$$\mathbb{I}_4^1 = \{1,2,3,4\}$$

$$P\left(\mathbb{I}_4^1\right) = \{\varnothing,1,2,3,4,12,13,14,23,24,34,123,234,134,124,1234\}$$

$$\Theta_C = \{\varnothing,1,2,3,4,13,24\}, \Psi_C = \{12,23,34,14,123,234,134,124,1234\}$$

$$\Psi_C(12) = \{23,34,14,234,134\}, \Psi_C(123) = \{34,14,134\}, \Psi_C(1234) = \{\ \}$$

$$\mathbb{R}_C(1,\varnothing) = p_4^4, \mathbb{R}_C(1,1) = \mathbb{R}_C(1,2) = \mathbb{R}_C(1,3) = \mathbb{R}_C(1,4) = p_4^1,$$

$$\mathbb{R}_C(1,13) = \mathbb{R}_C(1,24) = p_4^2$$

$$\mathbb{R}_C(1,12) = \mathbb{R}_C(1,23) = \mathbb{R}_C(1,34) = \mathbb{R}_C(1,14) = p_4^2$$

$$\mathbb{R}_C(1,123) = \mathbb{R}_C(1,234) = \mathbb{R}_C(1,134) = \mathbb{R}_C(1,124) = p_4^1, \mathbb{R}_C(1,1234) = p_4^0$$

$$\mathbb{R}_C(1,\Theta) = \sum_{Y\in\Theta_C} \mathbb{R}_C(1,Y) = p_4^4 + 4p_4^3 + 2p_4^2$$

$$\mathbb{R}_C(1,\Psi) = \sum_{Y\in\Psi_C} \mathbb{R}_C(1,Y) = 4p_4^2 + 4p_4^1 + p_4^0$$

$$\mathbb{R}_C(1) = \mathbb{R}_C(1,\Theta) + R_C(1,\Psi) = p_4^4 + 4p_4^3 + 6p_4^2 + 4p_4^1 + p_4^0$$

$$\mathbb{R}_C(2,\Theta) = \mathbb{R}_C(1,\Theta)\mathbb{R}_C(1) = p_8^8 + 8p_8^7 + 24p_8^6 + 36p_8^5 + 29p_8^4 + 12p_8^3 + 2p_8^2$$

$$\mathbb{R}_C(2,12) = \mathbb{R}_C(2,23) = \mathbb{R}_C(2,34) = \mathbb{R}_C(2,14)$$

$$\mathbb{R}_C(2,12) = R(12)\left[\mathbb{R}_C(1,\Theta) + \sum_{Y\in\Psi_C(12)} \mathbb{R}_C(1,Y)\right]$$

$$= R(12)\left[\mathbb{R}_C(1,\Theta) + \mathbb{R}_C(1,23) + \mathbb{R}_C(1,34) + \mathbb{R}_C(1,14) + \mathbb{R}_C(1,234) + \mathbb{R}_C(1,134)\right]$$

$$\mathbb{R}_C(2,12) = p_4^2\left[p_4^4 + 4p_4^3 + 2p_4^2 + 3p_4^2 + 2p_4^1\right] = p_8^6 + 4p_8^5 + 5p_8^4 + 2p_8^3$$

$$\mathbb{R}_C(2,123) = \mathbb{R}_C(2,234) = \mathbb{R}_C(2,134) = \mathbb{R}_C(2,124)$$

$$\mathbb{R}_C(2,123) = R(123)\left[\mathbb{R}_C(1,\Theta) + \sum_{Y\in\Psi_C(123)} \mathbb{R}_C(1,Y)\right]$$

$$= R(123)\left[\mathbb{R}_C(1,\Theta) + \mathbb{R}_C(1,34) + \mathbb{R}_C(1,14) + \mathbb{R}_C(1,134)\right]$$

$$= p_4^1\left[p_4^4 + 4p_4^3 + 2p_4^2 + 2p_4^2 + p_4^1\right] = p_8^5 + 4p_8^4 + 4p_8^3 + p_8^2$$

$$\mathbb{R}_C(2,1234) = p_4^0\left[\mathbb{R}_C(1,\Theta) + \underbrace{\sum_{Y\in\Psi_C(1234)} \mathbb{R}_C(1,Y)}_{0}\right]$$

$$= p_4^0\left[p_4^4 + 4p_4^3 + 2p_4^2\right] = p_8^4 + 4p_8^3 + 2p_8^2$$

$$\mathbb{R}_C(2,\Psi) = \sum_{Y\in\Psi_C} \mathbb{R}_C(2,Y) = 4\mathbb{R}_C(2,12) + 4\mathbb{R}_C(2,123) + \mathbb{R}_C(2,1234)$$

$$= 4p_8^6 + 20p_8^5 + 37p_8^4 + 28p_8^3 + 6p_8^2$$

$$\mathbb{R}_C(2) = \mathbb{R}_C(2,\Theta) + \mathbb{R}_C(2,\Psi)$$

$$= p_8^8 + 8p_8^7 + 28p_8^6 + 56p_8^5 + 66p_8^4 + 40p_8^3 + 8p_8^2$$

$$\mathbb{R}_C(3,\Theta) = \mathbb{R}_C(2)\mathbb{R}_C(1,\Theta)$$

$$= p_{12}^{12} + 12p_{12}^{11} + 62p_{12}^{10} + 184p_{12}^9 + 346p_{12}^8 + 416p_{12}^7 + 300p_{12}^6 + 112p_{12}^5 + 16p_{12}^4$$

$$\mathbb{R}_C(3,12) = \mathbb{R}_C(3,23) = \mathbb{R}_C(3,34) = \mathbb{R}_C(3,14)$$

$$\mathbb{R}_C(3,12) = R(12)\left[\mathbb{R}_C(2,\Theta) + \sum_{Y\in\Psi_C(12)} \mathbb{R}_C(2,Y)\right]$$

$$= R(12)\left[\mathbb{R}_C(2,\Theta) + \mathbb{R}_C(2,23) + \mathbb{R}_C(2,34) + \mathbb{R}_C(2,14) + \mathbb{R}_C(2,234) + \mathbb{R}_C(2,134)\right]$$

$$= p_{12}^{10} + 8p_{12}^9 + 27p_{12}^8 + 50p_{12}^7 + 52p_{12}^6 + 26p_{12}^5 + 4p_{12}^4$$

$$\mathbb{R}_C(3,123) = \mathbb{R}_C(3,234) = \mathbb{R}_C(3,134) = \mathbb{R}_C(3,124)$$

$$= R(123)\left[\mathbb{R}_C(2,\Theta) + \sum_{Y\in\Psi_C(123)} \mathbb{R}_{L(C)}(2,Y)\right]$$

$$= R(123)\left[\mathbb{R}_C(2,\Theta) + \mathbb{R}_C(2,34) + \mathbb{R}_C(2,14) + \mathbb{R}_C(2,134)\right]$$

$$= p_{12}^9 + 8p_{12}^8 + 26p_{12}^7 + 45p_{12}^6 + 43p_{12}^5 + 20p_{12}^4 + 3p_{12}^3$$

$$\mathbb{R}_C\left(3,1234\right)=R\left(1234\right)\left[\mathbb{R}_C\left(2,\Theta\right)+\underbrace{\sum_{Y\in\Psi_C(1234)}\mathbb{R}_C\left(2,Y\right)}_{0}\right]$$

$$=p_{12}^{8}+8p_{12}^{7}+24p_{12}^{6}+36p_{12}^{5}+29p_{12}^{4}+12p_{12}^{3}+2p_{12}^{2}$$

$$\mathbb{R}_C\left(3,\Psi\right)=\sum_{Y\in\Psi_C}\mathbb{R}_C\left(3,Y\right)=4\mathbb{R}_C\left(3,12\right)+4\mathbb{R}_C\left(3,123\right)+\mathbb{R}_C\left(3,1234\right)$$

$$=4p_{12}^{10}+36p_{12}^{9}+141p_{12}^{8}+312p_{12}^{7}+412p_{12}^{6}+312p_{12}^{5}+125p_{12}^{4}$$
$$+24p_{12}^{3}+2p_{12}^{2}$$

$$\mathbb{R}_C\left(3\right)=\mathbb{R}_C\left(3,\Theta\right)+\mathbb{R}_C\left(3,\Psi\right)$$

$$=p_{12}^{12}+12p_{12}^{11}+66p_{12}^{10}+220p_{12}^{9}+487p_{12}^{8}+728p_{12}^{7}+712p_{12}^{6}$$
$$+424p_{12}^{5}+141p_{12}^{4}+24p_{12}^{3}+2p_{12}^{2}$$
$$=p_{12}^{12}+12p^{11}q+66p^{10}q^{2}+220p^{9}q^{3}+487p^{8}q^{4}+728p^{7}q^{5}$$
$$+712p^{6}q^{6}+424p^{5}q^{7}+141p^{4}q^{8}+24p^{3}q^{9}+2p^{2}q^{10}$$

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