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On Certain Class of Analytic Functions with Negative Coefficients Defined by Fractional Derivative Operators

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Abstract: The present paper investigates a new class of analytic and univalent functions with negative coefficient in the unit disc U, involving certain fractional operators.Characterization and distortion theorems, and other interesting properties of this class of functions are studied. Further class preserving integral operator and some closed theorems for this class are also indicated.

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Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic and univalent in the unit disk $U = \{z : |z| < 1\}$.

A function $f(z) \in S$ is said to be starlike of order α $(0 \leq \alpha < 1)$, denoted by $S(\alpha)$, if and only if

$$Re\{\frac{zf'(z)}{f(z)}\} > \alpha \quad (z \in U), \tag{1.2}$$

and it is called convex of order α $(0 \leq \alpha < 1)$, denoted by $K(\alpha)$, if and only if

$$Re\{1 + \frac{zf''(z)}{f'(z)}\} > \alpha \quad (z \in U). \tag{1.3}$$

Let $P(\alpha)$ denote the class of functions $f(z) \in S$ such that

$$Re\{f'(z)\} > \alpha \ \ (0 \leq \alpha < 1) \ \ (z \in U). \tag{1.4}$$

If $f(z)$ given by (1.1)and $\phi(z) \in S$ is defined by

$$\phi(z) = z + \sum_{n=2}^{\infty} d_n \, z^n, \tag{1.5}$$

then the convolution or Hadamard product of $f(z)$and $\phi(z)$ is given by

$$(f * \phi)(z) = z + \sum_{n=2}^{\infty} a_n d_n \, z^n. \tag{1.6}$$

Let T denote the class of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (z \in U) \quad (a_n \geq 0), \tag{1.7}$$

which are analytic and univalent in U,and let

$$S^*(\alpha) = S(\alpha) \bigcap T \ , \ K^*(\alpha) = K(\alpha) \bigcap T \ , \ P^*(\alpha) = P(\alpha) \bigcap T. \tag{1.8}$$

These classes have been studied by Silverman[4],Gupta and Jain[1],Silverman and Silvia [5],and others.

A class of fractional derivative operator

Following Raina and Nahar [3](see also [2]),the fractional derivative operator$D_{0,z}^{\lambda,\mu,\eta}$ of a function $f(z)$ is defined as follows.
Definition 1. For $m - 1 \leq \lambda < m; m \in N \ and \ \mu, \eta \in R$

$$D_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{d^m}{dz^m}\left\{\frac{z^{\lambda-\mu}}{\Gamma(m-\lambda)} \int_0^z (z-t)^{m-\lambda-1} \right. \\ \left. {}_2F_1(\mu-\lambda, m-\eta; m-\lambda; 1-\tfrac{t}{z}) f(t)dt\right\}, \tag{2.1}$$

where the function $f(z)$ is analytic in a simply connected region of the z-plane containing the region , with the order

$$f(z) = o(|z|^r) \ , \ z \longrightarrow 0, \tag{2.2}$$

where $r > max\{0, \mu - \eta\} - 1$ and the multiplicity of $(z-t)^{m-\lambda-1}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$ and is well defined in the unit disk.

The operator defined by (2.1) includes the well known Riemann-Liouville fractional derivative operator ${}_0D_z^{\lambda} f(z)$ [9]. Indeed we have

$$D_{0,z}^{\lambda,\lambda,\eta} f(z) = {}_0D_z^{\lambda} f(z), \tag{2.3}$$

It is convenient to introduce here the fractional operator $J_{0.z}^{\lambda,\mu,\eta}$which is defined in terms of $D_{0,z}^{\lambda,\mu,\eta}$as follows.

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{\Gamma(2-\mu)\Gamma(2-\lambda+\eta)}{\Gamma(2-\mu+\eta)}\ z^{\mu}\ D_{0,z}^{\lambda,\mu,\eta} f(z), \tag{2.4}$$

$$(\lambda \geq 0\ ;\ \mu < 2\ ;\ \eta > max\{\lambda, \mu\} - 2)$$

It may noted that if $\lambda = \mu$ in (2.4), then by virtue of (2.3)we have

$$J_{0,z}^{\lambda,\lambda,\eta} f(z) = \Gamma(2-\lambda)\ z^{\lambda}\ {}_0D_z^{\lambda} f(z)\ , \tag{2.5}$$

and for $\lambda = \mu = 0$

$$J_{0,z}^{0,0,\eta} f(z) = f(z), \tag{2.6}$$

also for $\lambda = \mu = 1$

$$J_{o,z}^{1,1,\eta} f(z) = zf'(z). \tag{2.7}$$

With the aid of the operator (2.4) and the convolution technique we define the following class $P^*_{\lambda,\ \mu}(A, B, \alpha)$ of certain subclass of starlike and convex functions.

Definition 2. For A, B fixed,$-1 \leq B < A \leq 1$,$0 \leq \alpha < 1$,$\lambda \geq 0$,$\mu < 2$, and $\eta > max\{\lambda, \mu\} - 2$, let $P^*_{\lambda,\ \mu}(A, B, \alpha)$ denote the class of functions $f(z)$ of the form (1.7) such that

$$\frac{J_{0,z}^{\lambda,\mu,\eta}(f * h)(z)}{J_{0,z}^{\lambda,\mu,\eta}(f * g)(z)} \prec \frac{1 + [B + (A-B)(1-\alpha)]z}{1 + Bz}\ ,\ z \in U, \tag{2.8}$$

where $\prec$ denote the subordination and

$$g(z) = z + \sum_{n=2}^{\infty} b_n\ z^n\ ,\ h(z) = z + \sum_{n=2}^{\infty} c_n\ z^n,$$

with $c_n > b_n > 0$.

From the definition (2.8) it follows that $f(z) \in P^*_{\lambda,\ \mu}(A, B, \alpha)$ if there exist a function $w(z)$ regular in U and satisfying $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$ such that

$$\frac{J_{0,z}^{\lambda,\mu,\eta}(f*h)(z)}{J_{0,z}^{\lambda,\mu,\eta}(f*g)(z)} = \frac{1+[B+(A-B)(1-\alpha)]w(z)}{1+Bw(z)} \ , \ z \in U \tag{2.9}$$

The condition (2.9)is equivalent to

$$\left| \frac{\frac{J_{0,z}^{\lambda,\mu,\eta}(f*h)(z)}{J_{0,z}^{\lambda,\mu,\eta}(f*g)(z)}-1}{B+(A-B)(1-\alpha)-B\frac{J_{0,z}^{\lambda,\mu,\eta}(f*h)(z)}{J_{0,z}^{\lambda,\mu,\eta}(f*g)(z)}} \right| < 1 \ , \ z \in U \tag{2.10}$$

It may be noted that for $\lambda = \mu = 0, A = -B = 1, h(z) = z(1-z)^{-2} and\ g(z) = z(1-z)^{-1}$ the class $P^*_{\lambda,\ \mu}(A,B,\alpha)$ reduces to the class $S^*(\alpha)$ of starlike function while for $\lambda = \mu = 1, A = -B = 1, h(z) = z(1-z)^{-2} and\ g(z) = z(1-z)^{-1}$ the class $P^*_{\lambda,\ \mu}(A,B,\alpha)$ reduces to the class $K^*(\alpha)$ of convex function, also the class $P^*(\alpha)$ can be obtained from $P^*_{\lambda,\ \mu}(A,B,\alpha)$ by choosing $\lambda = \mu = 0, A = -B = 1, h(z) = z(1-z)^{-2} and\ g(z) = z$.
Moreover by specifying the values of λ, μ, A, B and proper choosing of the function $g(z)$ and $h(z)$ we obtain several other classes studied and introduced by various researchers such as Singh and Sohi[7],Silverman and Silvia[6],Owa and Aouf [10],and others.

Before starting and proving our main theorems ,we need the following lemma to be used in the sequel (cf.Raina and Nahar [3])

Lemma 1. if $\lambda \geq 0; n > max\{0, \mu - \eta\} - 1$, then

$$D_{0,z}^{\lambda,\mu,\eta} z^n = \frac{\Gamma(n+1)\Gamma(n-\mu+\eta+1)}{\Gamma(n-\mu+1)\Gamma(n-\lambda+\eta+1)} \ z^{n-\mu}. \tag{2.11}$$

Coefficient estimates

Theorem 1. A function $f(z)$ defined by (1.7) is in the class $P^*_{\lambda,\ \mu}(A,B,\alpha)$ if and only if

$$\sum_{n=2}^{\infty}\{(A-B)(1-\alpha)b_n + (1-B)(c_n - b_n)\}a_n\delta(n) \leq (A-B)(1-\alpha) \tag{3.1}$$

where

$$\delta(n) = \frac{(2)_{n-1}(2+\eta-\mu)_{n-1}}{(2-\mu)_{n-1}(2+\eta-\lambda)_{n-1}} \ (n \geq 2) \tag{3.2}$$

The result is sharp.

Proof. Applying lemma 1 and (2.4), we get

$$J_{0,z}^{\lambda,\mu,\eta}(f*h)(z)=z-\sum_{n=2}^{\infty}\delta(n)a_nc_n\ z^n\ ,a_n\geq 0,c_n\geq 0$$

and

$$J_{0,z}^{\lambda,\mu,\eta}(f*g)(z)=z-\sum_{n=2}^{\infty}\delta(n)a_nb_n\ z^n\ ,a_n\geq 0,b_n\geq 0$$

where $\delta(n)$ is given by (3.2).
Assuming that (3.1) holds and $|z|=1$.Then we have

$$|J_{0,z}^{\lambda,\mu,\eta}(f*h)(z)-J_{0,z}^{\lambda,\mu,\eta}(f*g)(z)|$$

$$-|\{B+(A-B)(1-\alpha)\}J_{0,z}^{\lambda,\mu,\eta}(f*g)(z)-BJ_{0,z}^{\lambda,\mu,\eta}(f*h)(z)|$$

$$=|-\sum_{n=2}^{\infty}(c_n-b_n)\delta(n)a_n\ z^n|-|(A-B)(1-\alpha)z$$

$$-\sum_{n=2}^{\infty}\{(A-B)(1-\alpha)b_n-(c_n-b_n)B\}\delta(n)a_n\ z^n|$$

$$\leq\sum_{n=2}^{\infty}\{(A-B)(1-\alpha)b_n+(1-B)(c_n-b_n)\}\delta(n)a_n-(A-B)(1-\alpha)$$

$$\leq 0,$$

hence by maximum modulus principle $f(z)\in P^*_{\lambda,\ \mu}(A,B,\alpha)$.

Conversely ,assume that $f(z)$ is in the class $P^*_{\lambda,\ \mu}(A,B,\alpha)$.then

$$\left|\frac{\frac{J_{0,z}^{\lambda,\mu,\eta}(f*h)(z)}{J_{0,z}^{\lambda,\mu,\eta}(f*g)(z)}-1}{B+(A-B)(1-\alpha)-B\frac{J_{0,z}^{\lambda,\mu,\eta}(f*h)(z)}{J_{0,z}^{\lambda,\mu,\eta}(f*g)(z)}}\right| \tag{3.3}$$

$$=\frac{|-\sum_{n=2}^{\infty}(c_n-b_n)\delta(n)a_n\ z^n|}{|(A-B)(1-\alpha)z-\sum_{n=2}^{\infty}\{(A-B)(1-\alpha)b_n-(c_n-b_n)B\}\delta(n)a_n\ z^n|}$$

$$<1.$$

Since $|Re(z)|\leq|z|$ for any z,we find from (3.3) that

$$(3.4)\qquad Re\left\{\frac{-\sum_{n=2}^{\infty}(c_n-b_n)\delta(n)a_n\ z^n}{(A-B)(1-\alpha)z-\sum_{n=2}^{\infty}\{(A-B)(1-\alpha)b_n-(c_n-b_n)B\}\delta(n)a_n\ z^n}\right\} < 1.$$

Now choosing the value of z on the real axis so that $\frac{J_{0,z}^{\lambda,\mu,\eta}(f*h)(z)}{J_{0,z}^{\lambda,\mu,\eta}(f*g)(z)}$is real, then upon clearing the denominator in (3.4) and letting $z \to 1$ through real values we have

$$\sum_{n=2}^{\infty}(c_n - b_n)\delta(n)a_n\ z^n \le (A-B)(1-\alpha)z - \sum_{n=2}^{\infty}\{(A-B)(1-\alpha)b_n - (c_n - b_n)B\}\delta(n)a_n\ z^n$$

which gives the desired assertion(3.1).

Finally ,we note that equality in (3.1) holds for the function

$$(3.5)\qquad f(z) = z - \frac{(A-B)(1-\alpha)}{\{(A-B)(1-\alpha)b_n+(1-B)(c_n-b_n)\}\ \delta(n)}\ z^n.$$

Note that the coefficient estimate theorem of the classes $S^*(\alpha)$ *and* $K^*(\alpha)$ due to Silverman and Chaterjee [8] can be obtained from Theorem 1 by setting respectively $\lambda = \mu = 0, A = -B = 1, c_n = n, b_n = 1$ *and* $\lambda = \mu = 1, A = -B = 1, c_n = n, b_n = 1$.
Further the class $P^*(\alpha)$ can be obtained from Theorem 1 by choosing $\lambda = \mu = 0, A = -B = 1, c_n = n, and\ b_n = 0$.

Characterization theorem

Theorem 2. Let $\lambda', \mu', \eta' \in R$ such that $\lambda' \ge 0, \mu' < 2, max\{\lambda', \mu'\}-2 < \eta' \le \frac{\lambda'(\mu'-3)}{\mu'}$, and Let the function $f(z)$ defined by (1.7) satisfies

$$(4.1)\qquad \sum_{n=2}^{\infty}\frac{\{(A-B)(1-\alpha)b_n + (1-B)(c_n - b_n)\}}{(A-B)(1-\alpha)}\, a_n\delta(n) \le \frac{(2-\mu')(2+\eta'-\lambda')}{2(2+\eta'-\mu')}.$$

Then $J_{0,z}^{\lambda',\mu',\eta'}f(z)$ belonging to the class$P_{\lambda,\ \mu}^{*}(A, B, \alpha)$.The equality is attained.

Proof.Using the definition of fractional operator $J_{0,z}^{\lambda',\mu',\eta'} f(z)$ as defined in (2.4),and lemma 1 in (1.7) we have

$$J_{0,z}^{\lambda',\mu',\eta'} f(z) = z - \sum_{n=2}^{\infty} \delta'(n) a_n\, z^n, \tag{4.2}$$

where

$$\delta'(n) = \frac{(2)_{n-1}(2+\eta'-\mu')_{n-1}}{(2-\mu')_{n-1}(2+\eta'-\lambda')_{n-1}}\ (n \geq 2). \tag{4.3}$$

Under the conditions stated in the theorem , we observe that the function $\delta'(n)$ is non-increasing ,that is , it satisfies the inequality $\delta'(n+1) \leq \delta'(n)$ for all $n \geq 2$, and thus we have

$$0 < \delta'(n) \leq \delta'(2) = \frac{2(2+\eta'-\mu')}{(2-\mu')(2+\eta'-\lambda')}\quad , \tag{4.4}$$

thus (4.1) and (4.4) yield

$$\sum_{n=2}^{\infty} \frac{\{(A-B)(1-\alpha)b_n + (1-B)(c_n - b_n)\}}{(A-B)(1-\alpha)}\ a_n \delta(n)\delta'(n) \tag{4.5}$$

$$\leq \delta'(2) \sum_{n=2}^{\infty} \frac{\{(A-B)(1-\alpha)b_n + (1-B)(c_n - b_n)\}}{(A-B)(1-\alpha)}\ a_n \delta(n)\ \leq 1$$

Hence by Theorem1 $J_{0,z}^{\lambda',\mu',\eta'} f(z) \in P^*_{\lambda,\ \mu}(A,B,\alpha)$.The equality in (4.1) is attained for the function $f(z)$ defined by (3.5).

Remark. When $A = -B = 1, c_n = n, b_n = 1, \lambda = \mu = 0$,then Theorem 2 gives the corresponding result due to Raina and Nahar [3,p.4,Theorem1]. Similarly when$A = -B = 1, c_n = n, b_n = 1, \lambda = \mu = 1$,in Theorem2 we get the corresponding result in [8].

Distortion theorem

Theorem 3. Let $\lambda', \mu', \eta' \in R$ such that $\lambda' \geq 0, \mu' < 2, max\{\lambda', \mu'\} - 2 < \eta' \leq \frac{\lambda'(\mu'-3)}{\mu'}$, and let the function $f(z)$ defined by (1.7) be in the class $P^*_{\lambda,\ \mu}(A,B,\alpha)$. Then

$$|D_{0,z}^{\lambda',\mu',\eta'} f(z)| \geq \frac{\Gamma(2-\mu'+\eta')}{\Gamma(2-\mu')\Gamma(2-\lambda'+\eta')} |z|^{1-\mu'} \tag{5.1}$$

$$\times \left\{ 1 - \frac{(A-B)(1-\alpha)(2-\mu'+\eta')(2-\lambda+\eta)(2-\mu)}{[(A-B)(1-\alpha)b_2 + (1-B)(c_2-b_2)](2-\mu+\eta)(2-\mu')(2-\lambda'+\eta')}\ |z| \right\},$$

and

$$(5.2)\quad |D_{0,z}^{\lambda',\mu',\eta'}f(z)| \leq \frac{\Gamma(2-\mu'+\eta')}{\Gamma(2-\mu')\Gamma(2-\lambda'+\eta')}|z|^{1-\mu'}$$

$$\times\left\{ 1+\frac{(A-B)(1-\alpha)(2-\mu'+\eta')(2-\lambda+\eta)(2-\mu)}{[(A-B)(1-\alpha)b_2+(1-B)(c_2-b_2)](2-\mu+\eta)(2-\mu')(2-\lambda'+\eta')}\ |z|\ \right\},$$

for $z \in U\ if \mu' \leq 1\ and\ z \in U - \{0\}\ if\ \mu' > 1$.
Equalities in (5.1)and (5.2)are attained by the function given by (3.5).

Proof. In view of Theorem 1, we have

$$(5.3)\qquad \sum_{n=2}^{\infty} a_n \leq \frac{(A-B)(1-\alpha)}{\{(A-B)(1-\alpha)b_2+(1-B)(c_2-b_2)\}\delta(2)}\ ,$$

where

$$\delta(2) = \frac{2(2+\eta-\mu)}{(2-\mu)(2+\eta-\mu)}.$$

Making use of (4.4) and (5.3) in (4.2) , we see that

$$(5.4)\qquad |J_{0,z}^{\lambda',\mu',\eta'}f(z)| \geq |z|-|z|^2\sum_{n=2}^{\infty} a_n\delta'(n)$$

$$\geq |z|-\delta'(2)|z|^2\sum_{n=2}^{\infty} a_n$$

$$\geq |z|-\frac{(A-B)(1-\alpha)\delta'(2)}{\{(A-B)(1-\alpha)b_2+(1-B)(c_2-b_2)\}\delta(2)}|z|^2, \qquad ,$$

which implies the assertion (5.1)of Theorem 3.
The assertion (5.2)can be proved similarly.

Corollary 1. Let the function $f(z)$ defined by (1.7) be in the class $P^*_{\lambda,\ \mu}(A, B, \alpha)$.Then

$$(5.5)\qquad |f(z)| \geq |z|\left(1-\frac{(A-B)(1-\alpha)}{[(A-B)(1-\alpha)b_2+(1-B)(c_2-b_2)]\delta(2)}|z|\ \right),$$

and

$$(5.6)\qquad |f(z)| \leq |z|$$

$$\left(1+\frac{(A-B)(1-\alpha)}{[(A-B)(1-\alpha)b_2+(1-B)(c_2-b_2)]\delta(2)}|z|\right),$$

for $z \in U$.The result is attained for the function $f(z)$ given by (3.5).

Proof. setting $\lambda' = \mu' = 0$ in Theorem 3 using the relationship (2.6) ,we get the result.

Corollary 2. Let the function $f(z)$ defined by (1.7) be in the class $P^*_{\lambda,\ \mu}(A, B, \alpha)$.Then

$$|f'(z)| \geq 1 - \frac{2(A-B)(1-\alpha)}{[(A-B)(1-\alpha)b_2+(1-B)(c_2-b_2)]\delta(2)}|z|, \tag{5.7}$$

and

$$|f'(z)| \leq 1 + \frac{2(A-B)(1-\alpha)}{[(A-B)(1-\alpha)b_2+(1-B)(c_2-b_2)]\delta(2)}|z|, \tag{5.8}$$

for $z \in U$.The result is attained for the function $f(z)$ given by (3.5).

Proof. setting $\lambda' = \mu' = 1$ in Theorem 3 and using the relationship (2.7), we get the result.

Remark 2. Putting $A = -B = 1, \lambda = \mu = 0, b_2 = 1$ *and* $c_2 = 2$ in Corollary 1 ,we obtain the corresponding result due to srivestava,Owa and Chaterjea [8] for the class $S^*(\alpha)$.Similarly for $A = -B = 1, \lambda = \mu = 1, b_2 = 1$ *and* $c_2 = 2$ in Corollary1 , we obtain the corresponding result in [6] for the class $K^*(\alpha)$.

Radius of convexity

Theorem 4. Let the function $f(z)$ defined by (1.7)be in the class $P^*_{\lambda,\ \mu}(A, B, \alpha)$.Then$f(z)$ is convex in the disk $|z| < r_1 = r_1(A, B, \alpha)$,where

$$r_1(A, B, \alpha) = inf_{n\geq 2} \left\{ \frac{[(A-B)(1-\alpha)b_n+(1-B)(c_n-b_n)]\delta(n)}{n^2(A-B)(1-\alpha)} \right\}^{\frac{1}{n-1}}. \tag{6.1}$$

The result is attained for the function $f(z)$ given by (3.5).

Proof. To establish the required result, it is sufficient to show that

$|\frac{zf''(z)}{f'(z)}| \leq 1 \ for|z| < 1$, or equivalently

$$\frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n|z|^{n-1}} \leq 1,$$

or

(6.2) $$\sum_{n=2}^{\infty} n^2 a_n|z|^{n-1} \leq 1,$$

as $f(z) \in P^*_{\lambda,\ \mu}(A, B, \alpha)$, we have from Theorem 1

(6.3) $$\sum_{n=2}^{\infty} \frac{\{(A-B)(1-\alpha)b_n+(1-B)(c_n-b_n)\}}{(A-B)(1-\alpha)} \ a_n\delta(n) \leq 1.$$

Thus (6.2) is true if

(6.4) $$n^2|z|^{n-1} \leq \frac{\{(A-B)(1-\alpha)b_n+(1-B)(c_n-b_n)\}}{(A-B)(1-\alpha)} \ \delta(n).$$

Setting $|z| = r_1$ in (6.4) and simplify we get the result.

Integral transforms

Theorem 5. Let the function $f(z)$ defined by (1.7) be in the class $P^*_{\lambda,\ \mu}(A, B, \alpha)$.Then the integral transforms

(7.1) $$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) \ dt \quad (c > -1)$$

belongs $P^*_{\lambda,\ \mu}(A, B, \alpha)$.

Proof.Using (1.7), and (7.1)we get

$$F(z) = z - \sum_{n=2}^{\infty} \frac{c+1}{c+n} \ a_n \ z^n$$

Therefore

$$\sum_{n=2}^{\infty} \{(A-B)(1-\alpha)b_n + (1-B)(c_n - b_n)\} \frac{c+1}{c+n} \ a_n\delta(n)$$

$$< \sum_{n=2}^{\infty} \{(A-B)(1-\alpha)b_n + (1-B)(c_n - b_n)\} \ a_n\delta(n)$$

$$\leq (A-B)(1-\alpha)$$

which implies that $F(z) \in P^*_{\lambda,\ \mu}(A, B, \alpha)$.

Closure properties

Theorem 6. Let the functions

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{nj}\ z^n, (j = 1, 2, ..., m),$$

be in the class $P^*_{\lambda,\ \mu}(A, B, \alpha)$.Then the function $h(z)$ defined by

$$h(z) = z - \sum_{n=2}^{\infty} e_n\ z^n,$$

also belongs to $P^*_{\lambda,\ \mu}(A, B, \alpha)$, *where* $e_n = \frac{1}{m}\sum_{j=2}^{m} a_{nj}\ , (a_{nj} \geq 0)$.

Proof. Since $f_j(z) \in P^*_{\lambda,\ \mu}(A, B, \alpha)$,it follows from Theorem 1 that

$$\sum_{n=2}^{\infty}\{(A-B)(1-\alpha)b_n+(1-B)(c_n-b_n)\}a_{nj}\delta(n) \leq (A-B)(1-\alpha), \tag{8.1}$$

Therefore

$$\begin{aligned}
&\sum_{n=2}^{\infty}\{(A-B)(1-\alpha)b_n + (1-B)(c_n - b_n)\}e_n\delta(n) \\
&= \sum_{n=2}^{\infty}\{(A-B)(1-\alpha)b_n + (1-B)(c_n - b_n)\}(\frac{1}{m}\sum_{j=2}^{m} a_{nj})\delta(n) \\
&\leq (A-B)(1-\alpha)
\end{aligned}$$

which shows that $h(z) \in P^*_{\lambda,\ \mu}(A, B, \alpha)$.This proves the theorem.

Theorem 7. Let $f_1(z) = z$ and

$$f_n(z) = z - \frac{(A-B)(1-\alpha)}{\{(A-B)(1-\alpha)b_n + (1-B)(c_n - b_n)\}\delta(n)} z^n, (n \geq 2) \tag{8.2}$$

Then $f(z) \in P^*_{\lambda,\ \mu}(A, B, \alpha)$ if and only if it can be expressed in the form

$$f(z) = \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n(z), \tag{8.3}$$

where $\lambda_n \geq 0$ *and* $\lambda_1 + \sum_{n=2}^{\infty} \lambda_n = 1$

Proof. Let (8.3)hold.Then by (8.2),we have

$$f(z) = z - \sum_{n=2}^{\infty} \frac{(A-B)(1-\alpha)}{\{(A-B)(1-\alpha)b_n + (1-B)(c_n - b_n)\}\delta(n)} \lambda_n \, z^n$$

Now

$$\sum_{n=2}^{\infty}\{(A-B)(1-\alpha)b_n + (1-B)(c_n - b_n)\}\delta(n)$$

$$\times \; \frac{(A-B)(1-\alpha)}{\{(A-B)(1-\alpha)b_n + (1-B)(c_n - b_n)\}\delta(n)} \lambda_n$$

$$= (A-B)(1-\alpha)\sum_{n=2}^{\infty}\lambda_n$$

$$\leq (A-B)(1-\alpha)$$

Hence by Theorem 1 $f(z) \in P^*_{\lambda,\,\mu}(A, B, \alpha)$.

Conversely, suppose $f(z) \in P^*_{\lambda,\,\mu}(A, B, \alpha)$. Since

$$a_n \leq \frac{(A-B)(1-\alpha)}{\{(A-B)(1-\alpha)b_n + (1-B)(c_n - b_n)\}\delta(n)} \;, \; n \geq 2$$

Setting $$\lambda_n = \frac{\{(A-B)(1-\alpha)b_n + (1-B)(c_n - b_n)\}\delta(n)}{(A-B)(1-\alpha)} a_n \,,$$

and $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n$, we get (8.3).This completes the proof of the theorem.

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