Journal of Al Azhar University-Gaza (Natural Sciences), (2006), Vol. 8, 135-146 ISSN 1810-6366

Certain Generalization of Prestarlike and Convex Functions Defined by Fractional Derivative Operators

Jamal. M. Shenan
Department of Mathematics , Alazhar University-Gaza
P.O.Box 1277,Gaza,Palestine.
e-mail:salimtariq@yahoo.com

Abstract: The present paper investigates a general class of analytic and univalent functions with negative coefficient in the unit disc U, involving certain operators of fractional calculus. Distortion theorem and other properties of this class of functions are studied.Further preserving integral operator and some closed theorems for this class are also mentioned.

(2000Mathematics Subject Classification:30C45 26A33)
keywords and phrases: Fractional derivative,analytic, univalent,and prestarlike functions.

Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$.

A function $f(z) \in S$ is said to be starlike of order α $(0 \leq \alpha < 1)$, denoted by $S(\alpha)$, if and only if

$$Re\{\tfrac{zf'(z)}{f(z)}\} > \alpha \quad (z \in U), \tag{1.2}$$

and it is called convex of order α $(0 \leq \alpha < 1)$, denoted by $K(\alpha)$, if and only if

$$Re\{1 + \tfrac{zf''(z)}{f'(z)}\} > \alpha \quad (z \in U). \tag{1.3}$$

Further,let $P(\alpha)$ denote the class of functions $f(z) \in S$ such that

$$Re\{f'(z)\} > \alpha \quad (0 \leq \alpha < 1) \quad (z \in U). \tag{1.4}$$

Let $f(z)$ be defined by (1.1),and

$$\phi(z) = z + \sum_{n=2}^{\infty} d_n \, z^n \ . \tag{1.5}$$

Then the convolution or hadamard product of $f(z)$and $\phi(z)$ is given by

$$(f*\phi)(z) = z + \sum_{n=2}^{\infty} a_n d_n\ z^n\ . \tag{1.6}$$

A function $f(z) \in S$ is said to be pre-starlike of order α $(0 \leq \alpha < 1)$, denoted by $R(\alpha)$, if and only if

$$f(z)*z(1-z)^{2\alpha-2} \in S(\alpha)\ . \tag{1.7}$$

Let T denote the class of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n\ \ (z \in U)\ \ (a_n \geq 0)\ , \tag{1.8}$$

which are analytic and univalent in U, and

$S^*(\alpha) = S(\alpha) \bigcap T$, $K^*(\alpha) = K(\alpha) \bigcap T$, $P^*(\alpha) = P(\alpha) \bigcap T$ *and* $R^*(\alpha) = R(\alpha) \bigcap T$.

These classes have been studied by Silverman[5],Gupta and Jain[2],Silverman and Silvia [6],and others.

among several interesting definitions of fractional derivative and fractional integral, given in literature (cf. [1], [3]), we find it to be convenient to to restrict ourselves to the following definition used by Owa [11](and also by Srivastava and Owa [9]).

Definition 1. The fractional integral of order λ for a function $f(z)$ is defined by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}}\, dt\ \ , \tag{1.9}$$

where $\lambda > 0, f(z)$ is analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$.

Definition 2. The fractional derivative of order λ for a function $f(z)$ is defined by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\lambda}}\, dt\ \ , \tag{1.10}$$

where $0 \leq \lambda < 1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{-\lambda}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$.

Definition 3. Under the hypothesis of Definition 2, the fractional derivative of order $n+\lambda$ of $f(z)$ is defined by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z), \ (0 \leq \lambda < 1, \ n \in N_0) \tag{1.11}$$

Following Owa and Srivastava [12], we introduce the linear operator Ω_z^{λ} defined by

$$\Omega_z^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) \ , \tag{1.12}$$

It may noted that $\Omega_z^0 f(z) = f(z)$ and $\Omega_z^1 f(z) = zf'(z)$.
With the implication of the operator (1.12) and by the convolution technique, we define the following class $P_\lambda^*(A,B,\alpha)$ of certain subclass of prestarlike and convex functions.

Definition 4. For A, B fixed, $-1 \leq B < A \leq 1$, $0 \leq \alpha < 1$, $0 < \lambda \leq 1$, let $P_\lambda^*(A,B,\alpha)$ denote the class of functions $f(z)$ of the form (1.8) such that

$$\frac{\Omega_z^{\lambda}(f*h)(z)}{\Omega_z^{\lambda-1}(f*g)(z)} \prec \frac{1+[B+(A-B)(1-\alpha)]z}{1+Bz} \ , \ z \in U \tag{1.13}$$

Where $\prec$ denote the subordination and

$$g(z) = z + \sum_{n=2}^{\infty} b_n \ z^n \ , \ h(z) = z + \sum_{n=2}^{\infty} c_n \ z^n, c_n > b_n \geq 0$$

Therefore , a function $f(z) \in T$ belongs to the class $P_\lambda^*(A,B,\alpha)$ if there exists a function $w(z)$ regular in U and satisfying $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$ such that

$$\frac{\Omega_z^{\lambda}(f*h)(z)}{\Omega_z^{\lambda-1}(f*g)(z)} = \frac{1+[B+(A-B)(1-\alpha)]w(z)}{1+Bw(z)} \ , \ z \in U. \tag{1.14}$$

The condition (1.14)is equivalent to

$$\left| \frac{\frac{\Omega_z^{\lambda}(f*h)(z)}{\Omega_z^{\lambda-1}(f*g)(z)} -1}{B+(A-B)(1-\alpha)-B\frac{\Omega_z^{\lambda}(f*h)(z)}{\Omega_z^{\lambda-1}(f*g)(z)}} \right| < 1 \ , \ z \in U \tag{1.15}$$

It may be noted that the class $P_\lambda^*(A,B,\alpha)$ is very general, it extends the classes of starlike, convex and pre-starlike functions by assigning specific values to $A, B, b_n, c_n, and\ \lambda$ .in what follow, we mention important subclasses of the class $P_\lambda^*(A,B,\alpha)$.
(i) For $\lambda = c_n = A = -B = 1, and\ b_n = 0$ in (1.15), we obtain the class $P^*(\alpha)$.
(ii) For $\lambda = c_n = b_n = A = -B = 1$ in (1.15), we obtain the class $S^*(\alpha)$ of starlike functions of order α.
(iii) For $\lambda = A = -B = 1, and\ b_n = c_n = n$, we obtain the class $K^*(\alpha)$ of convex functions of order α.
(iv) For$\lambda = A = -B = 1, and\ b_n = c_n = B(n,\alpha) = \frac{(2(1-\alpha))_{n-1}}{(n-1)!}$, we obtain the class $R^*(\alpha)$ of pre-starlike functions of order α.

Several other classes which are studied by various researchers such as Singl and Sohi[8],Silverman and Silva[7],Owa and Aouf [13],Shukla and Shukla [4], can be obtained from defined above class $P_\lambda^*(A,B,\alpha)$.

In the present paper, we have obtained sharp results, involving coefficient estimates, distortion theorem, radius of convexity ,class preserving integral operators, and closure theorem for the class $P_\lambda^*(A,B,\alpha)$.

Coefficient estimates

Theorem 1. A function $f(z)$,defined by (1.8), is in the class $P_\lambda^*(A,B,\alpha)$ if and only if

$$\begin{aligned} \sum_{n=2}^{\infty}\{(A-B)(1-\alpha)(2-\lambda)b_n+[(n-\lambda+1)c_n-(2-\lambda)b_n](1-B)\}a_n\delta(n) \\ \leq (A-B)(1-\alpha) \end{aligned} \tag{2.1}$$

where

$$\delta(n) = \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n-\lambda+2)} \tag{2.2}$$

The result is sharp.

Proof. From (1.10)and (1.12),

$$\Omega_z^{\lambda}(f*h)(z)=z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n-\lambda+1)}\ a_nc_n\ z^n\ ,c_n\geq 0$$

and

$$\Omega_z^{\lambda-1}(f*g)(z)=z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(3-\lambda)}{\Gamma(n-\lambda+2)}\ a_nb_n\ z^n\ ,b_n\geq 0.$$

Assuming that (2.1) holds and $|z|=1$, we have

$$\begin{aligned}&|\Omega_z^{\lambda}(f*h)(z)-\Omega_z^{\lambda-1}(f*g)(z)|\\&-|\{B+(A-B)(1-\alpha)\}\Omega_z^{\lambda-1}(f*g)(z)-B\Omega_z^{\lambda}(f*h)(z)|\\&\quad=|-\sum_{n=2}^{\infty}((n-\lambda+1)c_n-(2-\lambda)b_n)\delta(n)\ a_n\ z^n|-|(A-B)(1-\alpha)z\\&\quad-\sum_{n=2}^{\infty}\{(A-B)(1-\alpha)(2-\lambda)b_n-((n-\lambda+1)c_n-(2-\lambda)b_n)B\}\delta(n)a_n\ z^n|\\&\quad\leq\sum_{n=2}^{\infty}\{(A-B)(1-\alpha)(2-\lambda)b_n+[(n-\lambda+1)c_n-(2-\lambda)b_n](1-B)\}a_n\delta(n)\\&\quad-(A-B)(1-\alpha)\\&\quad\leq 0.\end{aligned}$$

Hence by maximum modulus principle $f(z)\in P_{\lambda}^{*}(A,B,\alpha)$.

Conversely, assume that $f(z)$ is in the class $P_{\lambda}^{*}(A,B,\alpha)$. Then

$$\left|\frac{\frac{\Omega_z^{\lambda}(f*h)(z)}{\Omega_z^{\lambda-1}(f*g)(z)}-1}{B+(A-B)(1-\alpha)-B\frac{\Omega_z^{\lambda}(f*h)(z)}{\Omega_z^{\lambda-1}(f*g)(z)}}\right| \tag{2.3}$$

$$=\frac{|-\sum_{n=2}^{\infty}((n-\lambda+1)c_n-(2-\lambda)b_n)\delta(n)a_nz^n|}{|(A-B)(1-\alpha)z-\sum_{n=2}^{\infty}\{(A-B)(1-\alpha)(2-\lambda)b_n-((n-\lambda+1)c_n-(2-\lambda)b_n)B\}\delta(n)a_nz^n|}$$

$<1.$

Since $|Re(z)| \leq |z|$ for any z,we find from (2.3) that

$$(2.4)\quad Re\left\{ \frac{-\sum_{n=2}^{\infty}((n-\lambda+1)c_n-(2-\lambda)b_n)\delta(n)a_nz^n}{(A-B)(1-\alpha)z-\sum_{n=2}^{\infty}\{(A-B)(1-\alpha)(2-\lambda)b_n-((n-\lambda+1)c_n-(2-\lambda)b_n)B\}\delta(n)a_nz^n} \right\}$$
$$< 1.$$

Now choosing, the value of z on the real axis so that $\frac{\Omega_z^{\lambda}(f*h)(z)}{\Omega_z^{\lambda-1}(f*g)(z)}$is real, then upon clearing the denominator in (2.4) and letting $z \to 1$ through real values, we have

$$\sum_{n=2}^{\infty}((n-\lambda+1)c_n-(2-\lambda)b_n)\delta(n)a_n \leq (A-B)(1-\alpha)$$

$$-\sum_{n=2}^{\infty}\{(A-B)(1-\alpha)(2-\lambda)b_n-((n-\lambda+1)c_n-(2-\lambda)b_n)B\}\delta(n)a_n$$

which gives the desired assertion(2.1).

Finally ,we note that equality in (2.1) holds for the function

$$(2.5)\quad f(z) = z-\frac{(A-B)(1-\alpha)}{\{(A-B)(1-\alpha)(2-\lambda)b_n+((n-\lambda+1)c_n-(2-\lambda)b_n)(1-B)\}\ \delta(n)}\, z^n.$$

Distortion theorem

Theorem 2. Let the function $f(z)$, defined by (1.8), be in the class $P_{\lambda}^{*}(A,B,\alpha)$. Then (3.1) $|D_z^{\lambda'-1}f(z)| \geq \frac{|z|^{2-\lambda'}}{\Gamma(3-\lambda')}$

$$\times \left\{ 1-\frac{(A-B)(1-\alpha)(2-\lambda)(3-\lambda)}{[(A-B)(1-\alpha)(2-\lambda)b_2+((3-\lambda)c_2-(2-\lambda)b_2)(1-B)](3-\lambda')}\ |z| \right\}$$

and

$$(3.2)\ |D_z^{\lambda'-1}f(z)| \leq \frac{|z|^{2-\lambda'}}{\Gamma(3-\lambda')}$$
$$\times \left\{ 1+\frac{(A-B)(1-\alpha)(2-\lambda)(3-\lambda)}{[(A-B)(1-\alpha)(2-\lambda)b_2+((3-\lambda)c_2-(2-\lambda)b_2)(1-B)](3-\lambda')}\ |z| \right\}$$

for $z \in U$ *and* $0 < \lambda' \leq 1$.
Equalities in (3.1)and (3.2)are attained by the function given by (2.5).

Proof. In view of Theorem 1, we have

$$(3.3)\sum_{n=2}^{\infty} a_n \leq \frac{(A-B)(1-\alpha)}{\{(A-B)(1-\alpha)(2-\lambda)b_2+((3-\lambda)c_2-(2-\lambda)b_2)(1-B)\}\delta(2)},$$

where

$$\delta(2)=\frac{2}{(2-\lambda)(3-\lambda)} \quad .$$

Next, by using the definition of fractional operator $\Omega_z^{\lambda'-1}f(z)$, as defined in (1.12), we have.

$$\Omega_z^{\lambda'-1}f(z)=z-\sum_{n=2}^{\infty}\delta'(n)a_n\,z^n \quad , \tag{3.4}$$

where

$$\delta'(n)=\frac{\Gamma(3-\lambda')(\Gamma(n+1))}{\Gamma(n-\lambda'+2)} \tag{3.5}$$

It is easily seen that $\delta'(n)$ is non-increasing ,that is , it satisfies the inequality $\delta'(n+1)\leq\delta'(n)$ for all $n\geq 2$, and we have

$$0<\delta'(n)\leq\delta'(2)=\frac{2}{(3-\lambda')} \quad . \tag{3.6}$$

Consequently, we obtain

$$|\Omega_z^{\lambda'-1}f(z)|\geq|z|-|z|^2\sum_{n=2}^{\infty}a_n\delta'(n) \tag{3.7}$$

$$\geq|z|-\delta'(2)|z|^2\sum_{n=2}^{\infty}a_n$$

$$\geq|z|-\frac{(A-B)(1-\alpha)\delta'(2)}{\{(A-B)(1-\alpha)(2-\lambda)b_2+((3-\lambda)c_2-(2-\lambda)b_2)(1-B)\}\delta(2)}|z|^2$$

which proves (3.1).
similarly (3.2) can be proved and thus, details are avoided.

Corollary . Let the function $f(z)$, defined by (1.8), be in the class $P_\lambda^*(A,B,\alpha)$.Then

$$(3.8)|f(z)|\geq|z|\left(1-\frac{(A-B)(1-\alpha)(2-\lambda)(3-\lambda)}{2[(A-B)(1-\alpha)(2-\lambda)b_2+((3-\lambda)c_2-(2-\lambda)b_2)(1-B)]}|z|\right)$$

and

(3.9) $|f(z)| \leq |z| \left(1+\frac{(A-B)(1-\alpha)(2-\lambda)(3-\lambda)}{2[(A-B)(1-\alpha)(2-\lambda)b_2+((3-\lambda)c_2-(2-\lambda)b_2)(1-B)]}|z|\right)$

the result is sharp for the function defined in (2.5).

Proof. The proof follows readily from Theorem 2 in the special case when $\lambda' = 1$.

Remark . Putting $A = -B = \lambda = b_2 = c_2 = 1$ in the Corollary 1 ,we obtain the corresponding result due to Srivastava Owa and chaterjea [10] for the class $S^*(\alpha)$.Similarly, for $A = -B = \lambda = 1, b_2 = c_2 = 2$, we obtain the corresponding result in [4] for the class $K^*(\alpha)$.

Radius of convexity

Theorem 3. Let the function $f(z)$,defined by (1.8), be in the class $P^*_\lambda(A, B, \alpha)$, then$f(z)$ is convex in the disc $|z| < r$,where

(4.1) $r = inf_{\ n\geq 2} \left\{ \frac{[(A-B)(1-\alpha)(2-\lambda)b_n+((n-\lambda+1)c_n-(2-\lambda)b_n)\}(1-B)]\delta(n)}{n^2(A-B)(1-\alpha)} \right\}^{\frac{1}{n-1}}$

The result is sharp for the function $f(z)$ given by (2.5).

Proof. To establish the required result it is sufficient to show that

$|\frac{zf''(z)}{f'(z)}| \leq 1\ for|z| < 1$, or equivalently

$$\frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n|z|^{n-1}} \leq 1$$

or

(4.2) $$\sum_{n=2}^{\infty} n^2 a_n|z|^{n-1} \leq 1$$

as $f(z) \in P^*_{\lambda,\ \mu}(A, B, \alpha)$, we have from Theorem 1

(4.3)$\sum_{n=2}^{\infty} \frac{\{(A-B)(1-\alpha)(2-\lambda)b_n+((n-\lambda+1)c_n-(2-\lambda)b_n)(1-B)\}}{(A-B)(1-\alpha)} a_n\delta(n) \leq 1$

Thus, (4.2) is true if

(4.4) $$n^2|z|^{n-1} \leq \frac{\{(A-B)(1-\alpha)(2-\lambda)b_n+((n-\lambda+1)c_n-(2-\lambda)b_n)(1-B)\}}{(A-B)(1-\alpha)}\,\delta(n)$$

Setting $|z| = r_1$ in (4.4) and on simplification, the required result is obtained.

Integral transforms

Theorem 4. Let the function $f(z)$, defined by (1.8), be in the class $P^*_\lambda(A, B, \alpha)$,then the integral transforms

(5.1) $$F(z) = \tfrac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt \quad (c > -1)$$

belongs $P^*_\lambda(A, B, \alpha)$.

Proof. From (1.8) and (5.1),we get

$$F(z) = z - \sum_{n=2}^{\infty} \frac{c+1}{c+n}\, a_n\, z^n \ .$$

Therefor,

$$\sum_{n=2}^{\infty}\{(A-B)(1-\alpha)(2-\lambda)b_n+((n-\lambda+1)c_n-(2-\lambda)b_n)(1-B)\}\frac{c+1}{c+n}\, a_n\delta(n)$$

$$< \sum_{n=2}^{\infty}\{(A-B)(1-\alpha)(2-\lambda)b_n+((n-\lambda+1)c_n-(2-\lambda)b_n(1-B))\}\, a_n\delta(n)$$
$$\leq (A-B)(1-\alpha) \ ,$$

which implies that $F(z) \in P^*_\lambda(A, B, \alpha)$.This completes the proof of the theorem.

Closure Properties

Theorem 5. Let the functions

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{nj}\, z^n, (j = 1, 2, ..., m)$$

be in the class $P^*_\lambda(A, B, \alpha)$. Then the function $h(z)$ defined by

$$h(z) = z - \sum_{n=2}^{\infty} e_n\, z^n$$

also belongs to $P_\lambda^*(A,B,\alpha)$, *where* $e_n = \dfrac{1}{m}\sum_{n=2}^{m} a_{nj}$, $(a_{nj} \geq 0)$.

Proof. Since $f_j(z) \in P_\lambda^*(A,B,\alpha)$,it follows from Theorem 1, that

(6.1) $\sum_{n=2}^{\infty}\{(A-B)(1-\alpha)(2-\lambda)b_n + ((n-\lambda+1)c_n - (2-\lambda)b_n)(1-B)\}a_{nj}\delta(n)$

$$\leq (A-B)(1-\alpha)$$

Therefor,

$$\sum_{n=2}^{\infty}\{(A-B)(1-\alpha)(2-\lambda)b_n+((n-\lambda+1)c_n-(2-\lambda)b_n)(1-B)\}e_n\delta(n)$$

$$= \sum_{n=2}^{\infty}\{(A-B)(1-\alpha)(2-\lambda)b_n+((n-\lambda+1)c_n-(2-\lambda)b_n)(1-B)\}(\frac{1}{m}\sum_{n=2}^{m} a_{nj})\delta(n)$$

$\leq (A-B)(1-\alpha)$, by (6.1),
which shows that $h(z) \in P_\lambda^*(A,B,\alpha)$.This completes the proof of the theorem.

Theorem 6. Let $f_1(z) = z$ and

$$(6.2) \quad f_n(z) = z - \frac{(A-B)(1-\alpha)}{\{(A-B)(1-\alpha)(2-\lambda)b_n+((n-\lambda+1)c_n-(2-\lambda)b_n)(1-B)\}\delta(n)}\, z^n$$

Then $f(z) \in P_\lambda^*(A,B,\alpha)$ if and only if it can be expressed in the form

$$(6.3) \qquad f(z) = \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n(z) \ ,$$

where $\lambda_n \geq 0$ *and* $\lambda_1 + \sum_{n=2}^{\infty} \lambda_n = 1$

Proof. Let (6.3)holds true ,then by (6.2),we have

$$f(z) = z - \sum_{n=2}^{\infty} \frac{(A-B)(1-\alpha)}{\{(A-B)(1-\alpha)(2-\lambda)b_n+((n-\lambda+1)c_n-(2-\lambda)b_n)(1-B)\}\delta(n)}\, \lambda_n\, z^n$$

Now,

$$\sum_{n=2}^{\infty}\{(A-B)(1-\alpha)b_n+((n-\lambda+1)c_n-(2-\lambda)b_n)(1-B)\}\delta(n)$$

$$\times \frac{(A-B)(1-\alpha)}{\{(A-B)(1-\alpha)(2-\lambda)b_n+((n-\lambda+1)c_n-(2-\lambda)b_n)(1-B)\}\delta(n)}\,\lambda_n$$
$$= (A-B)(1-\alpha)\sum_{n=2}^{\infty}\lambda_n$$
$$\leq (A-B)(1-\alpha)$$

Hence, by Theorem1 $f(z) \in P_{\lambda}^{*}(A,B,\alpha)$.

Conversely, suppose $f(z) \in P_{\lambda}^{*}(A,B,\alpha)$. Since

$$a_n \leq \frac{(A-B)(1-\alpha)}{\{(A-B)(1-\alpha)(2-\lambda)b_n+((n-\lambda+1)c_n-(2-\lambda)b_n)(1-B)\}\delta(n)}$$

Setting

$$\lambda_n = \frac{\{(A-B)(1-\alpha)(2-\lambda)b_n+((n-\lambda+1)c_n-(2-\lambda)b_n)(1-B)\}\delta(n)}{(A-B)(1-\alpha)}\,a_n\,,$$

and $\lambda_1 = 1-\sum_{n=2}^{\infty}\lambda_n$, we get (6.3).This completes the proof of the theorem.

References

1 Erdelyi, A. Magnus, W. Oberhettinger, F. and Tricomi, F. G. Tables of integral transformations, 1954, Vol II, MacGraw-hill, New York/Toronto
/London.

2 Gupta V. P. and jain, P. K. Certain classes of univalent functions with negative coefficient II, Bull. Austral. Math. Soc.1976, (15), 467-473.

3 Saigo, M. A remark on integral operators in Gauss hypergeometric functions, Math. Rep. College Gen. Edu. Kyushu Univ.1978, 11, (2), 135-143.

4 Shukla, N.and Shukla, P. On a class of analytic functions defined by fractional derivative, II, Kyungpook Math. J.1998, (38), 3821-28.

5 Silverman, H. Univalent functions with negative coefficient, Proc. Amer. Math. Soc.1975, (51), 109-116.

6 Silverman, H. and Silvia, E. M. Prestarlike functions with negative coefficient, Internat J. Math. Sci.1979, (2), 427-439.

7 Silverman, H. and Silvia, E. M. Subclasses of starlike functions subordinate to convex functions. Canad. J. Math. 1985 , 37-48.

8 Singh, A. and Sohi, N. S. On certain subclasses of univalent functions, Pure. Appl. Math. Sci.1995, (61), 1-2 .

9 Srivastava, H. M. and Owa, S. An application of fractional derivative , Math. Japon.1984, (29), 383-389.

10 Srivastava, H. M. Owa, S. and Chatarj, S. K. A note on certain class of starlike functions, Rend. Sem. Math. Univ. Padova, 1987, (77), 115-124.

11 Owa, S. On the distortion theorems I, Kyungpook Math. J. 1978, (18), 53-59.

12 Owa, S. and Srivastava, H. M. Univalent and starlike generalized hypergeometric functions, Canad. J. Math.1987, 39(5), 1057-1077.

13 Owa, S. and Aouf, M. K. On subclasses of univalent functions with negative coefficients, Pure Appl. Math. Sci.1989, (29) , 131-139.