Conditions Of Univalence, Starlikeness And Convexity Of The *H*-Function

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Abstract *In this paper, several results concerning the univalence, starlikeness and convexity in certain classes of H-function have been established. These results are applicable to special functions like Meijer's G-function, Wright's hypergeometric function and the generalized hypergeometric function. Some of these special cases are mentioned briefly.*

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INTRODUCTION

Let $m, n, p, q \in N_0 = N \cup \{0\}$ such that $0 \le m \le q$ $, 0 \le n \le p$ and let

$a_j, b_j \in C$ and $A_j, B_j \in R_+ = (0, \infty)$ $(j = 1,...q;\ \ i = 1,...p)$. The *H*-function

occurring in this paper is defined by ([1,p.408])

$$H^{m,n}_{p,q}[z] = H^{m,n}_{p,q}\left[z \left| \begin{matrix} (a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,q} \end{matrix} \right. \right] = \frac{1}{2\pi i}\int_L \theta(s) z^s ds \qquad (1.1)$$

where

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - B_j s)\prod_{j=1}^{n}\Gamma(1 - a_j + A_j s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + B_j s)\prod_{j=n+1}^{p}\Gamma(a_j - A_j s)} \qquad (1.2)$$

The contour L is specially chosen and an empty product, if it occurs, is set as one. The theory of this function can be found in [2,Chap.1], [3,8.3] and [4,Chap.2].

A special case of the *H*-function (1.1) can be written as:

$$h(z) = H^{1,n}_{p,q+1}\left[-z \left| \begin{matrix} (a_j, A_j)_{1,p} \\ (0,1),(b_j, B_j)_{1,q} \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{n}\Gamma(1-a_j+A_j s)\Gamma(-s)}{\prod_{j=1}^{q}\Gamma(1-b_j+B_j s)\prod_{j=n+1}^{p}\Gamma(a_j-A_j s)}(-z)^s\,ds \qquad (1.3)$$

which can be written in series form as [4,p.12, Eq. (2.24)]

$$h(z) = \sum_{r=0}^{\infty} c_r z^r \qquad (1.4)$$

where $c_r = \dfrac{\prod_{j=1}^{n}\Gamma(1-a_j+A_j r)}{\prod_{j=1}^{q}\Gamma(1-b_j+B_j r)\prod_{j=n+1}^{p}\Gamma(a_j-A_j r)}\dfrac{1}{r!}$

and $A_j r \neq a_j - k - 1$ $(1 \le j \le n;\ k, r \in N_0)$.

Denote by E the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty}\alpha_n z^n \qquad (1.5)$$

which are analytic and univalent in the unit disk $u = \{z : |z| < 1\}$.

Let *S(A,B)* denote the subclass of functions $f(z) \in E$ satisfying the inequality

$$\left|\frac{[zf'(z)/f(z)]-1}{A-B[zf'(z)/f(z)]}\right| < 1 \qquad (z \in u) \qquad (1.6)$$

where $-1 \le B < A \le 1$ and $-1 \le B \le 0$.

Also, let *K(A, B)* denote the subclass of functions $f(z) \in E$ such that $zf'(z) \in S(A,B)$. The functions belonging to *S(A, B)* are starlike of order *(A+B)/2B* and type $|B|$. Further, the functions belonging to *K(A, B)* are said to be convex.

Setting $A = 1-2\alpha$ and B =-1 and $S(1-2\alpha,-1) = S^*(\alpha)$ in (1.6), then $S^*(\alpha)$ represents the class of functions $f(z) \in E$ which are starlike of order $\alpha (0 \le \alpha < 1)$ satisfying

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in u) \qquad (1.7)$$

Also, if we put $K(1-2\alpha,-1) = K^*(\alpha)$, then $K^*(\alpha)$ represents the class of functions $f(z) \in E$ which are convex of order α satisfying

$$1 + \mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in u) \qquad (1.8)$$

where $0 \le \alpha < 1$. It is being understood that the functions such as $zf'(z)/f(z)$ which have removable singularities at *z=0*, have had these singularities removed throughout this paper.

For general references to the aforementioned definitions and statements, one can refer to [5], [6] and [7].

UNIVALENT *H*-FUNCTION

A function $f(z) \in E$ is said to be close-to-convex if there is a convex function *p(z)* such that

$$\operatorname{Re}\left(\frac{f'(z)}{p'(z)}\right) > 0 \qquad (z \in u) \tag{2.1}$$

In the sequel the following two lemmas due to Jack [8] and Duran [6], respectively are needed

Lemma 1: *If w(z) is analytic in the unit disk u such that w(0)=0 and* $|w(z_1)| = \max_{|z|=r} |w(z)|, \quad 0 \le r < 1$, *then*

$$z_1 w'(z_1) = k w(z_1) \quad , k \ge 1 \tag{2.2}$$

Lemma 2: *Every close-to-convex function is univalent.*

Now we prove the first result on univalent *H*-function contained in

Theorem 1: *Let the H-function h(z) defined by (1.4) satisfy the condition*

$$|h'(z) - c_1|^{1-\alpha} \left|\frac{zh''(z)}{h'(z)}\right|^{\alpha} < (c_1)^{1-\alpha} (\tfrac{1}{2})^{\alpha} \tag{2.3}$$

for some fixed $\alpha \ge 0$ *where* $z \in u$ *and,* $c_0 > 0$, $c_1 > 0$ *.Then h(z) is univalent in the unit disk u.*

Proof: Consider the function

$$\phi(z) = \frac{h(z) - c_0}{c_1}, \tag{2.4}$$

where $z \in u$ .Then $\phi(z) \in E$ and (2.3) implies

$$\left|\phi'(z)-1\right|^{1-\alpha}\left|z\frac{\phi''(z)}{\phi'(z)}\right|^{\alpha}<\left(\frac{1}{2}\right)^{\alpha} \tag{2.5}$$

Now, let

$$w(z)=\phi'(z)-1 \qquad (z\in u) \tag{2.6}$$

Which is analytic in the unit disk *u* and *w(0)=0* since $\phi'(0)=1$ (which can be easily verified from (2.4) in conjunction with the definition (1.4)). So, (2.5) can be written as

. $$\left|w(z)\right|\left|\frac{zw'(z)}{w(z)}\cdot\frac{1}{1+w(z)}\right|^{\alpha}<\left(\frac{1}{2}\right)^{\alpha} \tag{2.7}$$

where the comment about removable singularities applies just as in (1.7). Assume that there exists a point $z_1 \in u$ such that

$$\max_{|z|\le|z_1|}\left|w(z)\right|=\left|w(z_1)\right|=1\ , \tag{2.8}$$

then, we can put $z_1(w'(z_1)/w(z_1))=k\ \ (k\ge 1)$ by means of Lemma1. Therefore we get

$$\left|w(z_1)\right|\left|\frac{z_1w'(z_1)}{w(z_1)}\cdot\frac{1}{1+w(z_1)}\right|^{\alpha}\ge\left(\frac{k}{2}\right)^{\alpha}\ge\left(\frac{1}{2}\right)^{\alpha} \tag{2.9}$$

which contradicts the assertions (2.7) and (2.3), hence $\left|w(z)\right|=\left|\phi'(z)-1\right|<1$ which implies that $\mathrm{Re}(\phi'(z))>0$ and which means at the same time that $\mathrm{Re}(h'(z))>0$.

Now, since *p(z)=z* is convex in the unit disk *u*, then

$$\mathrm{Re}\left(\frac{\phi'(z)}{p'(z)}\right)>0 \qquad (z\in u)$$

And so *h(z)* is close-to-convex in *u*. This means, finally that *h(z)* is univalent in *u* (in view of Lemma 2)

Setting $\alpha=1$ and $\alpha=0$, respectively, then Theorem 1 yields the following:

Corollary 1: *Let h(z) be defined by (1.4), such that*

$$\left|\frac{zh''(z)}{h'(z)}\right|<\frac{1}{2} \qquad (z\in u) \tag{2.10}$$

where $c_0>0$ *and* $c_1>0$. *Then h(z) is univalent in u.*

Corollary 2: *Let h(z) be defined by (1.4), such that*

$$\left|h'(z)-c_1\right|<c_1 \tag{2.11}$$

where $c_0 > 0$ *and* $c_1 > 0$. *Then h(z) is univalent in u.*

STARLIKE AND CONVEX H-FUNCTION

The following Lemma due to Raina [5,Lemma 3, p.16] is required

Lemma 3: *Let f(z) be defined by (1.5) satisfying*

$$\left|\frac{zf'(z)}{f(z)}-1\right|^{1-\alpha}\left|\frac{zf''(z)}{f'(z)}\right|^{\alpha} < \frac{(A-B)(2+A+A^2)^{\alpha}}{(1+|B|)(1+A)^{2\alpha}} \tag{3.1}$$

for fixed constants A,B and α *such that* $-1 \le B < A < 1,\ -1 \le B \le 0$ *and*

$\alpha \ge 0,\ \ \forall z \in u$. *Then* $f(z) \in S(A,B)$.

Now, we apply Lemma 3 in order to prove the following theorems concerning the starlikeness of the *H*-function.

Theorem 2*: Let h(z) defined by (1.4) satisfies*

$$\left|\frac{zh'(z)}{h(z)}\right| < \frac{A-B}{1+|B|} \tag{3.2}$$

for $-1 \le B < A < 1$ *and* $-1 \le B \le 0$, *then*

$$\frac{z}{c_0}h(z) \in S(A,B) \tag{3.3}$$

Proof: Define the function *F(z)* by

$$F(z) = \frac{z}{c_0}h(z), \qquad (z \in u) \tag{3.4}$$

Then (3.2) yields

$$\left|\frac{zF'(z)}{F(z)}-1\right| < \frac{A-B}{1+|B|} \tag{3.5}$$

So , applying (3.1) with $\alpha = 0$, we get the result required.

Theorem 3: *Let h(z) defined by (1.4) satisfies*

$$\left|\frac{zh''(z)}{h'(z)}\right| < \frac{(A-B)(2+A+A^2)}{(1+|B|)(1+A)^2} \tag{3.6}$$

for $-1 \le B < A < 1$ *and* $-1 \le B \le 0$, *then h(z) is starlike of order (A+B)/2B and type* $|B|$ *with respect to* c_0.

Proof: The function $\phi(z)$ defined by (2.4) is in the class E satisfying

$$\left|\frac{z\phi''(z)}{\phi'(z)}\right| = \left|\frac{zh''(z)}{h'(z)}\right| < \frac{(A-B)(2+A+A^2)}{(1+|B|)(1+A)^2} \quad , \qquad (z \in u) \qquad (3.7)$$

so, by virtue of Lemma 3 with $\alpha = 1$, we conclude that $\phi(z) \in S(A,B)$, hence $\phi(z)$ is starlike of order *(A+B)/2B* and type $|B|$ with respect to the origin. This implies that *h(z)* is starlike of order *(A+B)/2B* and type $|B|$ with respect to c_0.

Now we establish a result on convex *H*-function contained in

Theorem 4: *Let h(z) defined by (1.4) satisfies the condition (3.2).Then*

$$\frac{z}{c_0}.H^{1,n+1}_{p+1,q+2}\left[-z\left|\begin{matrix}(0,1),(a_j,A_j)_{1,p}\\(0,1),(b_j,B_j)_{1,q},(-1,1)\end{matrix}\right.\right] \in K(A,B) \qquad (3.8)$$

Proof: Since $zf'(z) \in S(A,B) \Leftrightarrow f(z) \in K(A,B)$,therefore condition (3.2) implies

$$\frac{z}{c_0}.h(z) \in S(A,B) \;\Rightarrow \frac{z}{c_0}.\int_0^z h(t)dt \in K(A,B)$$

which yields the required result.

Due to the generality of the *H*-function defined by (1.4) , the results obtained by Theorems 1-4 can be made applicable to various special functions . For example , setting $A_i = 1$ and $B_j = 1$ $(i = 1,..., p \;\; ; j = 1,...q)$ in (1.4) , then

$$H^{1,n}_{p,q+1}\left[-z\left|\begin{matrix}(a_j,1)_{1,p}\\(0,1),(b_j,1)_{1,q}\end{matrix}\right.\right] = \sum_{r=0}^{\infty} e_r.z^r$$

$$= G^{1,n}_{p,q+1}\left[-z\left|\begin{matrix}(a_p)\\0,(b_q)\end{matrix}\right.\right] = g(z) \qquad (3.9)$$

where $e_r = \dfrac{\prod_{j=1}^{n}\Gamma(1-a_j+r)}{\prod_{j=1}^{q}\Gamma(1-b_j+r)\prod_{j=n+1}^{p}\Gamma(a_j-r)}\dfrac{1}{r!}$.

So, from Theorem 1 , we can get the following result

Corollary 3***:*** *Let g(z) be defined by (3.9) such that*

$$\left|g'(z)-e_1\right|^{1-\alpha}\left|\frac{zg''(z)}{g'(z)}\right|^{\alpha} < (e_1)^{1-\alpha}\left(\tfrac{1}{2}\right)^{\alpha} \qquad (3.10)$$

for fixed $\alpha \geq 0$ *where* $z \in u$ *and,* $e_0 < 0$, $e_1 < 0$ *.Then g(z) is univalent in the unit disk u.*

The other results concerning starlikeness and convexity of the *G*-function can be obtained easily from Theorems 2,3 and 4 . On the other hand setting *n=p* in (1.4) , we get

$$H^{1,p}_{p,q+1}\left[-z\left|\begin{matrix}(a_j,A_j)_{1,p}\\(0,1),(b_j,B_j)_{1,q}\end{matrix}\right.\right] = {}_p\Psi_q\left[\begin{matrix}(1-a_1,A_1),\ldots,(1-a_p,A_p);\\(1-b_1,B_1),\ldots,(1-b_q,B_q);\end{matrix}\; z\right] \qquad (3.11)$$

where ${}_p\Psi_q[z]$ is the Wright's generalized hypergeometric function [9] , and all the results obtained by Theorems 1-4 are equivalent to that recently obtained by Raina [5, Theorems 1-4] . Also ${}_p\Psi_q[z]$ contains as special cases the Mittag-Leffler function $E_{\alpha,\beta}[z]$ and the Bessel-Maitland function $J^{\mu}_{\nu}(z)$, so the results obtained by Raina [5, corollaries 3-9] are also special cases of the present results.

Setting *n=p* in (3.9) , we get

$$G^{1,p}_{p,q+1}\left[-z\left|\begin{matrix}(a_p)\\0,(b_q)\end{matrix}\right.\right] = \frac{\prod_{j=1}^{p}\Gamma(1-a_j)}{\prod_{j=1}^{q}\Gamma(1-b_j)}\; {}_pF_q\left[\begin{matrix}(1-a_1),\ldots,(1-a_p);\\(1-b_1),\ldots,(1-b_q);\end{matrix}\; z\right] \qquad (3.12)$$

where ${}_pF_q[z]$ is the generalized hypergeometric function [10]. So by virtue of (3.12), Theorems 1-4 correspond to the results obtained by Owa and Srivastava [11].

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