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On Some Integral Inequalities of Gronwall Type

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Abstract: In this paper, some integral inequalities of Gronwall type in more than one variable are presented. Some applications are presented also.

Keywords: Gronwall integral inequalities, Two and three independent variables, Hyperbolic equations .

INTRODUCTION

Inequalities play an important role in almost all branches of mathematics. The integral inequalities involving functions of one and more than one independent variables play a fundamental role in the study of differential and integral equations. By means of which, various investigators provided explicit bounds on the unknown functions, [1, 2, 3, 4, 5] and the references given therein.

In this paper, we establish explicit bounds on some certain integral inequalities of *Gronwall* type .

MAIN RESULTS

In what follows , R denotes the set of real numbers and $R_+ = [0,\infty)$, is the subset of R, $I_1 = [x_0, x)$, $I_2 = [y_0, y)$, *are the given subsets* of R_+, and $D = I_1 \times I_2$. The first order partial derivatives of a function $Z(x, y)$ defined for $x, y \in R$ with respect to x and y are denoted by $Z_x(x, y)$ and $Z_y(x, y)$ respectively. We use summation convention and throughout this paper, all the functions and their partial derivatives which appear in the inequalities are assumed to be real-valued and all the integrals involved are of positive values and exist on the respective domains of their definitions.

Our main results are given in the following theorems .

Theorem 1.
Let $u(x,y), h(x,y)$ *be nonnegative continuous functions and* $k(x,y) > 1$, *defined on* D *for* $x_0 \le s \le x,\ \ y_0 \le t \le y$. *Suppose that* $k_x(x,y), k_y(x,y)$, *and* $k_{xy}(x,y)$ *be nonnegative and continuous functions defined for* $x, y \in R_+$. *If*

$$u(x,y) \le k(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)u(s,t)dtds, \qquad (1)$$

for $x, y, x_0, y_0 \in R_+$, *then*

$$u(x,y) \le \frac{k(x_0,y)k(x,y_0)}{k(x_0,y_0)} \exp\{k(x,y) - k(x_0,y) - k(x,y_0) + k(x_0,y_0) + \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)dtds\}. \qquad (2)$$

Proof : Define a function $Z(x,y)$ on D by the right hand side of (1)

$$Z(x,y) = k(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)u(s,t)dtds, \qquad (3)$$

then $Z(x, y) > 1$. From (1), (3) we have

$$u(x,y) \le Z(x,y) \quad \text{for } x, y \in D \qquad (4)$$

Differentiating (3) with respect to x ,and then with respect to y we get

$$Z_x(x,y) = k_x(x,y) + \int_{y_0}^{y} h(x,t)u(x,t)dt ,$$

$$Z_{xy}(x,y) = k_{xy}(x,y) + h(x,y)u(x,y) \quad . \qquad (5)$$

From (4) and (5) we have

$$Z_{xy}(x,y) \le k_{xy}(x,y) + h(x,y)\ Z(x,y) \quad .$$

From which, we get

$$\frac{Z_{xy}(x,y)}{Z(x,y)} \le \frac{k_{xy}(x,y)}{Z(x,y)} + h(x,y) \quad . \tag{6}$$

Since $k(x,y) > 1$, then from (3) and (6), we have

$$\frac{Z_{xy}(x,y)}{Z(x,y)} < k_{xy}(x,y) + h(x,y) \tag{7}$$

and since $Z_x(x,y) \ge 0$ and $Z_y(x,y) \ge 0$, it is easy to see that

$$\frac{\partial}{\partial y}\left(\frac{Z_x}{Z}\right) \le \frac{Z_{xy}}{Z} \quad ,$$

so from (7) we have

$$\frac{\partial}{\partial y}\left(\frac{Z_x(x,y)}{Z(x,y)}\right) \le k_{xy}(x,y) + h(x,y) \quad . \tag{8}$$

Integrating (8) with respect to y from y_0 to y , we get

$$\frac{Z_x(x,y)}{Z(x,y)} - \frac{Z_x(x,y_0)}{Z(x,y_0)} \le k_x(x,y) - k_x(x,y_0) + \int_{y_0}^{y} h(x,t)dt \, ,$$

and hence

$$\frac{Z_x(x,y)}{Z(x,y)} \le \frac{Z_x(x,y_0)}{Z(x,y_0)} + k_x(x,y) - k_x(x,y_0) + \int_{y_0}^{y} h(x,t)dt \quad . \tag{9}$$

Integrating (9) with respect to x from x_0 to x , we obtain

$$\ln\frac{Z(x,y)}{Z(x_0,y)} \le \ln\frac{Z(x,y_0)}{Z(x_0,y_0)} + k\,(x,y) - k(x_0,y) - k(x,y_0) + k(x_0,y_0) + \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)dtds \tag{10}$$

From (3) , $Z(x,y_0) = k(x,y_0), Z(x_0,y) = k(x_0,y), Z(x_0,y_0) = k(x_0,y_0)$,
then we get from (10) that

$$\ln\left\{ \frac{Z(x,y)}{k(x_0,y)} \times \frac{k(x_0,y_0)}{k(x,y_0)} \right\} \leq k\ (x,y) - k(x_0,y) - k(x,y_0) + k(x_0,y_0) + \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)dtds,$$

from which, we have

$$Z(x,y) \leq \frac{k(x_0,y)k(x,y_0)}{k(x_0,y_0)} \exp\left\{ k\ (x,y) - k(x_0,y) - k(x,y_0) + k(x_0,y_0) + \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)dtds \right\}. \quad (11)$$

Hence, from (11) and (4) we have the required result .

The following special cases conclude four results which can be obtained by suitable substitutions in Theorem 1 .

<u>Special Cases I</u> .

1) If we replace $k(x,y)$ by a constant $c > 1$, then from Theorem1 , we have, if

$$u(x,y) \leq c + \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)u(s,t)dtds,$$

then

$$u(x,y) \leq c\exp\left(\int_{x_0}^{x}\int_{y_0}^{y} h(s,t)dtds\right). \quad (12)$$

2) If we replace $k(x,y)$ by a function of one variable $k(x)$, then from Theorem 1, we have , if

$$u(x,y) \leq k(x) + \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)u(s,t)dtds,$$

and k(x) > 1, then

$$u(x,y) \leq k(x)\exp\left(\int_{x_0}^{x}\int_{y_0}^{y} h(s,t)dtds\right). \quad (13)$$

3) If we replace $k(x,y)$ by $k(y)$, then from Theorem 1, we have, if

$$u(x,y) \leq k(y) + \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)u(s,t)dtds,$$

then

$$u(x,y) \leq k(y)\exp\left(\int_{x_0}^{x}\int_{y_0}^{y} h(s,t)dtds\right). \quad (14)$$

where $k\ (x, y_0) = k(y_0),\ k\ (x_0, y) = k(y),\ k(x_0, y_0) = k(y_0)$.

4) If we replace $k(x,y)$ by $c + k(x) + k(y)$, then from Theorem 1, we have, if

$$u(x, y) \le (c + k(x) + k(y)) + \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)u(s,t)dtds,$$

then

$$u(x,y) \le \left(\frac{[c+k(x)+k(y_0)][c+k(x_0)+k(y)]}{c+k(x_0)+k(y_0)} \right) \exp\left(\int_{x_0}^{x}\int_{y_0}^{y} h(s,t)dtds\right) \quad (15)$$

where

$$k(x, y_0) = c + k(x) + k(y_0), k(x_0, y) = c + k(x_0) + k(y),\ k(x_0, y_0) = c + k(x_0) + k(y_0).$$

We note that the proofs of (1)-(4) can be completed by following the proof of Theorem 1 with suitable changes .

In the following theorem, we consider functions of three independent variables .

Theorem 2.
Let $u(x, y, z), h(x, y, z)$ be nonnegative continuous functions and

$k(x, y, z) > 1$, defined for $x, y, z \in R_+$. Suppose that the derivatives

$k_x(x, y, z)$, $k_y(x, y, z)$, $k_{xy}(x, y, z)$, $k_{xyz}(x, y, z)$, be nonnegative and continuous functions defined for $x, y, z \in R_+$. If

$$u(x, y, z) \le k(x, y, z) + \int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} h(r, s, t,)u(r, s, t)dtdsdr \qquad (16)$$

for $x, y, z, x_0, y_0, z_0 \in R_+$, *then*

$$u(x,y,z) \le \frac{k(x_0,y,z)k(x,y_0,z)k(x,y,z_0)k(x_0,y_0,z_0)}{k(x_0,y_0,z)k(x_0,y,z_0)k(x,y_0,z_0)} \exp\Big\{ k(x,y,z) - k(x_0,y,z) - k(x,y_0,z) + k(x_0,y_0,z) - k(x,y,z_0) + k(x_0,y,z_0) + k(x,y_0,z_0) - k(x_0,y_0,z_0) + \int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} h(r,s,t)dtdsdr \Big\}. \tag{17}$$

Proof : Define a function $W(x,y,z)$ by the right hand side of (16),

$$W(x,y,z) = k(x,y,z) + \int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} h(r,s,t,)u(r,s,t)dtdsdr, \tag{18}$$

then

$W(x,y,z) > 0$ for $x,y,z \in R_+$, and from (16), (18), we have

$$u(x,y,z) \le W(x,y,z) \tag{19}$$

Differentiating (18) with respect to x, with respect to y and then with respect to z, respectively gives

$$W_x(x,y,z) = k_x(x,y,z) + \int_{y_0}^{y}\int_{z_0}^{z} h(x,s,t)u(x,s,t)dtds, \tag{20}$$

$$W_{xy}(x,y,z) = k_{xy}(x,y,z) + \int_{z_0}^{z} h(x,y,t)u(x,y,t)dt, \tag{21}$$

$$W_{xyz}(x,y,z) = k_{xyz}(x,y,z) + h(x,y,z)u(x,y,z). \tag{22}$$

From (19) and (22), we have

$$W_{xyz}(x,y,z) \le k_{xyz}(x,y,z) + h(x,y,z)W(x,y,z). \tag{23}$$

and from (23), we get

$$\frac{W_{xyz}(x,y,z)}{W(x,y,z)} \leq \frac{k_{xyz}(x,y,z)}{W(x,y,z)} + h(x,y,z). \quad (24)$$

Since $k(x,y,z) \geq 1$, we have

$$\frac{W_{xyz}(x,y,z)}{W(x,y,z)} \leq k_{xyz}(x,y,z) + h(x,y,z) \quad . \quad (25)$$

Since $W_z(x,y,z) > 0$ and $W_{xy}(x,y,z) > 0$, it easy to see that

$$\frac{\partial}{\partial z}\left(\frac{W_{xy}}{W}\right) \leq \frac{W_{xyz}}{W} \quad , \quad (26)$$

then from (25) and (26) , we obtain

$$\frac{\partial}{\partial z}\left(\frac{W_{xy}(x,y,z)}{W(x,y,z)}\right) \leq k_{xyz}(x,y,z) + h(x,y,z) \ . \quad (27)$$

Integrating (27) with respect to z from z_0 *to* z, we get

$$\frac{W_{xy}(x,y,z)}{W(x,y,z)} - \frac{W_{xy}(x,y,z_0)}{W(x,y,z_0)} \leq k_{xy}(x,y,z) - k_{xy}(x,y,z_0) + \int_{z_0}^{z} h(x,y,t)dt \quad (28)$$

Now from (21) and (18) we have

$$W_{xy}(x,y,z_0) = k_{xy}(x,y,z_0) \ , \ W\ (x,y,z_0) = k(x,y,z_0) \, ,$$

hence, from (28), we obtain

$$\frac{W_{xy}(x,y,z)}{W(x,y,z)} - \frac{k_{xy}(x,y,z_0)}{k(x,y,z_0)} \leq K_{xy}(x,y,z) - K_{xy}(x,y,z_0) + \int_{z_0}^{z} h(x,y,t)dt. \quad (29)$$

Since

$$\frac{\partial}{\partial y}\left[\frac{W_x(x,y,z)}{W(x,y,z)}-\frac{k_x(x,y,z_0)}{k(x,y,z_0)}\right]=\left[\frac{W_{xy}(x,y,z)}{W(x,y,z)}-\frac{k_{xy}(x,y,z_0)}{k(x,y,z_0)}\right]-$$
$$\left[\frac{W_x(x,y,z)W_y(x,y,z)}{W^2(x,y,z)}-\frac{k_x(x,y,z_0)k_y(x,y,z_0)}{k^2(x,y,z_0)}\right], \tag{30}$$

and
$$\frac{W_x(x,y,z)W_y(x,y,z)}{W^2(x,y,z)}\geq\frac{k_x(x,y,z_0)k_y(x,y,z_0)}{k^2(x,y,z_0)}, \tag{31}$$

we get from (30) , (31), that

$$\frac{\partial}{\partial y}\left(\frac{W_x(x,y,z)}{W(x,y,z)}-\frac{k_x(x,y,z_0)}{k\ \ (x,y,z_0)}\right)\leq\frac{W_{xy}(x,y,z)}{W(x,y,z)}-\frac{k_{xy}(x,y,z_0)}{k(x,y,z_0)}, \tag{32}$$

Now from (29) and (32), we obtain

$$\frac{\partial}{\partial y}\left(\frac{W_x(x,y,z)}{W(x,y,z)}-\frac{k_x(x,y,z_0)}{k(x,y,z_0)}\right)\leq k_{xy}(x,y,z)-k_{xy}(x,y,z_0)+\int_{z_0}^{z} h(x,y,t)dt. \tag{33}$$

Integrating (33) with respect to y from y_0 *to* y, we have

$$\frac{W_x(x,y,z)}{W(x,y,z)}-\frac{W_x(x,y_0,z)}{W(x,y_0,z)}\leq\frac{k_x(x,y,z_0)}{k(x,y,z_0)}-\frac{k_x(x,y_0,z_0)}{k(x,y_0,z_0)}+k_x(x,y,z)-k_x(x,y_0,z\ )$$

$$-k_x(x,y,z_0)+k_x(x,y_0,z_0)+\int_{y_0}^{y}\int_{z_0}^{z} h(x,s,t)dtds \tag{34}$$

From (20),

$$W_x(x,y_0,z)=k_x(x,y_0,z\ ),\ \ W(x,y_0,z)=k(x,y_0,z),$$

from which and (34), we get

$$\frac{W_x(x,y,z)}{W(x,y,z)}-\frac{k_x(x,y_0,z)}{k(x,y_0,z)}\le\frac{k_x(x,y,z_0)}{k(x,y,z_0)}-\frac{k_x(x,y_0,z_0)}{k(x,y_0,z_0)}+k_x(x,y,z)-k_x(x,y_0,z)-$$

$$k_x(x,y,z_0)+k_x(x,y_0,z_0)+\int_{y_0}^{y}\int_{z_0}^{z}h(x,s,t)dtds \tag{35}$$

Integrating (35) with respect to x from x_0 *to* x , we have

$$\ln\left(\frac{W(x,y,z)}{W(x_0,y,z)}\times\frac{k(x_0,y_0,z)}{k(x,y_0,z)}\right)-\ln\left(\frac{k(x,y,z_0)}{k(x_0,y,z_0)}\times\frac{k(x_0,y_0,z_0)}{k(x,y_0,z_0)}\right)$$

$$\le k(x,y,z)-k(x_0,y,z)-k(x,y_0,z)+k(x_0,y_0,z)-k(x,y,z_0)+$$

$$k(x_0,y,z_0)+k(x,y_0,z_0)-k(x_0,y_0,z_0)+\int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z}h(r,s,t)dtdsdr, \tag{36}$$

from which we obtain

$$W(x,y,z)\le\frac{k(x_0,y,z)k(x,y_0,z)k(x,y,z_0)k(x_0,y_0,z_0)}{k(x_0,y_0,z)k(x_0,y,z_0)k(x,y_0,z_0)}\exp\{\ k(x,y,z)-$$

$$k(x_0,y,z)-k(x,y_0,z)+k(x_0,y_0,z)-k(x,y,z_0)+k(x_0,y,z_0)+$$

$$k(x,y_0,z_0)-k(x_0,y_0,z_0)+\int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z}h(r,s,t)dtdsdr\ \}. \tag{37}$$

Hence, from (37) and (19), we have the required result .

The following special cases conclude five results which can be obtained by suitable substitutions in Theorem 2 .

<u>Special Cases II</u> .

1) If we replace $k(x,y,z)$ by a constant $c > 1$, then from Theorem 2 , we have, if

$$u(x,y,z) \le c + \int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} h(r,s,t)u(r,s,t)dtdsdr,$$

then

$$u(x,y,z) \le c\ \exp\left(\int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} h(r,s,t)dtdsdr\right). \tag{38}$$

2) If we replace $k(x, y, z)$ by $k(x) > 1$, then from Theorem 2, we have, if

$$u(x,y,z) \le k(x) + \int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} h(r,s,t)u(r,s,t)dtdsdr,$$

then

$$u(x,y,z) \le k(x)\exp\left(\int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} h(r,s,t)dtdsdr\right). \tag{39}$$

3) If we replace $k(x, y, z)$ by $k(y)>1$, then from Theorem 2, we have, if

$$u(x,y,z) \le k(y) + \int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} h(r,s,t)u(r,s,t)dtdsdr,$$

then

$$u(x,y,z) \le k(y)\exp\left(\int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} h(r,s,t)dtdsdr\right). \tag{40}$$

4) If we replace $k(x,y,z)$ by $c+ k(x)+ k(y)$, then from Theorem 2, we have, if

$$u(x,y,z) \le (c + k(x) + k(y)) + \int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} h(r,s,t)u(r,s,t)dtdsdr,$$

then

$$u(x,y,z) \le (c + k(x) + k(y))\exp\left(\int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} h(r,s,t)dtdsdr\right). \tag{41}$$

5) If we replace $k(x,y,z)$ by $k(x,y)$, then from Theorem 2, we have, if

$$u(x,y,z) \le k(x,y) + \int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} h(r,s,t)u(r,s,t)dtdsdr,$$

then

$$u(x,y,z) \le k(x,y)\exp\left(\int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} h(r,s,t)dtdsdr\right). \tag{42}$$

We note that the proofs of (1)-(5) can be completed by following the proof of Theorem 2 with suitable changes .

APPLICATIONS

In this section we present some applications of Theorem 1 and Theorem 2. We aim to study certain qualitative properties of solutions of hyperbolic partial differential equations .

Example 1.

Let $u_{xy}(x,y) = h(x,y)u(x,y)$, where

$$u_x(x,y_0) = 0\,,\ u(x_0,y) = c,\ c > 1,\ h(x,y) \le xy,$$

then

$$u_{xy}(x,y) \le xyu(x,y) \tag{43}$$

Integrating (43) with respect to y from y_0 *to* y, and with respect to x from x_0 *to* x we get

$$u(x,y) \le c + \int_{x_0}^{x}\int_{y_0}^{y} stu(s,t)dtds,$$

then by Theorem 1 with $k(x,y) = c$,

$$u(x,y) \le c\exp\left(\frac{1}{4}(x^2 - x_0^2)(y^2 - y_0^2)\right).$$

Example 2.

Let $k(x, y) = \frac{1}{2}(x^2 - x_0^2) = k(x)$, for $\frac{1}{2}(x^2 - x_0^2) > 1$. Consider the equation

$$u_{xy}(x,y) = h(x,y)u(x,y),$$

where

$$u_x(x, y_0) = x,\ u(x_0, y) = 0,\quad h(x,y\ ) \le ye^x,$$

then
$$u_{xy}(x,y) \le ye^x u(x,y) \tag{44}$$

Integrating (44) with respect to y from y_0 *to* y, and with respect to x from x_0 *to* x we get

$$u(x,y) \le \frac{1}{2}(x^2 - x_0^2) + \int_{x_0}^{x}\int_{y_0}^{y} te^s u(s,t)dtds,$$

then by Theorem 1,

$$u(x,y) \le \frac{1}{2}(x^2 - x_0^2)\exp\left(\frac{1}{2}(y^2 - y_0^2)(e^x - e^{x_0})\right),\ \text{for}\ y \ge 0, x > \sqrt{2 + x_0^2}\ .$$

Example 3.

Let $k(x, y) = c + k(x) + k(y)$, where

$$c = \frac{1}{2}(x_0^2 - y_0^2),\ \ k(x) = \frac{1}{2}x^2, k(y) = \frac{1}{2}y^2,$$

and $\frac{1}{2}(x_0^2 + y_0^2) + \frac{1}{2}(x^2 + y^2) > 1$. Consider the equation

$$u_{xy}(x,y) = h(x,y)u(x,y),$$

where

$$u_x(x, y_0) = x,\ \ u(x_0, y) = x_0^2 + \frac{1}{2}y_0^2 + \frac{1}{2}y^2,\ h(x,y\ ) \le xy\ \ \text{then}$$

$$u_{xy}(x,y) \le xyu(x,y)\quad . \tag{45}$$

Integrating (45) with respect to y from y_0 *to* y, and with respect to x from x_0 *to* x we have

$$u(x,y) \le \frac{1}{2}(x_0^2 + y_0^2) + \frac{1}{2}(x^2 + y^2) + \int_{x_0}^{x}\int_{y_0}^{y} stu(s,t)dtds,$$

then by Theorem 1,

$$u(x,y) \le \frac{(\frac{1}{2}x_0^2 + y_0^2 + \frac{1}{2}x^2)(x_0^2 + \frac{1}{2}y_0^2 + \frac{1}{2}y^2)}{(x_0^2 + y_0^2)} \exp\left(\frac{1}{4}(x^2 - x_0^2)(y^2 - y_0^2)\right),$$

$x_0, y_0 \in (0,\infty)$, $x^2 + y^2 > 2 - x_0^2 - y_0^2$.

Example 4.

Let $k(x, y) = \frac{1}{4}(x^2 + x_0^2)y^2$, for $(x^2 + x_0^2)y^2 > 4$. Consider the equation $u_{xy}(x,y) = h(x,y)u(x,y) + xy$, where $h(x,y) \le x + y$, $u_x(x,y_0) = \frac{1}{2}xy_0^2$, $u(x_0,y) = \frac{1}{2}x_0^2y^2$, we have

$$u_{xy}(x,y) \le (x+y)u(x,y) + xy. \tag{46}$$

Integrating (46) with respect to y from y_0 *to* y, and with respect to x from x_0 *to* x,

we have

$$u(x,y) \le \frac{1}{4}(x^2 + x_0^2)y^2 + \int_{x_0}^{x}\int_{y_0}^{y} (s+t)u(s,t)dtds,$$

then by Theorem 1,

$$u(x,y) \le \frac{1}{4}(x^2 + x_0^2)y^2 \exp\left(k(x,y) - k(x_0,y) - k(x,y_0) + k(x_0,y_0) + \int_{x_0}^{x}\int_{y_0}^{y} (s+t)dtds\right),$$

and then

$$u(x,y) \le \frac{1}{4}(x^2 + x_0^2)y^2 \exp\left((x^2 - x_0^2)(y - y_0)(y + y_0 + 1) + (y^2 - y_0^2)(x - x_0)(x + x_0 + 1)\right).$$

Example 5.

Let $u_{xyz}(x,y,z) = h(x,y,z)u(x,y,z) + 8xyz,$ where $h(x,y,z) \le 2(x+y+z),$ $u_{xy}(x,y,z_0) = 4xyz_0^2$, $u_x(x,y_0,z) = 2xz^2 y_0^2$, $u(x_0,y,z) = x_0^2 z^2 y^2$, then

$$u_{xyz}(x,y,z) \le 2(x+y+z)u(x,y,z) + 8xyz. \tag{47}$$

Integrating (47) with respect to z from z_0 *to* z, with respect to y from y_0 *to* y, and with respect to *x* from x_0 *to* x we have

$$u(x,y,z) \le x^2y^2z^2 + \int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} 2(r+s+t)u(r,s,t)dtdsdr,$$

then by Theorem 2 with $k(x, y, z) = x^2y^2z^2$, for $x^2y^2z^2 > 1$,

$$u(x,y,z) \le x^2y^2z^2 \exp\Big(k(x,y,z) - k(x_0,y,z) - k(x,y_0,z) + k(x_0,y_0,z) -$$

$$k(x,y,z_0) + k(x_0,y,z_0) + k(x,y_0,z_0) - k(x_0,y_0,z_0) +$$

$$\int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} 2(r+s+t)dtdsdr \Big),$$

so

$$u(x,y,z) \le x^2y^2z^2 \exp\Big[(x^2-x_0^2)(y^2-y_0^2)(z^2-z_0^2) +$$

$$(x-x_0)(y-y_0)(z-z_0)(x+y+z-x_0-y_0-z_0) \Big].$$

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