HAMILTON-JACOBI METHOD ANALYSIS OF CONSTRAINED SYSTEM S AND DIRAC'S CONJECTURE

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Abstract *The Hamilton-Jacobi method to investigate singular systems is analyzed. This method leads us to obtain the set of Hamilton-Jacobi partial differential equations in many variables. If the system is integrable, then we obtain the correct canonical generalized equations of motion contrary to the Dirac's method, where the Dirac's conjecture is invalid. Four examples are studied.*

Key words: *Hamilton-Jacobi approach, Dirac's approach, higher order Lagrangians, Nother theorem.*

INTRODUCTION

Symmetry is now a fundamental concept in modern physics (Castellani 1982, Gogilidze et al 1994, Li 1991, Li 1994 and Li 1999). The connection between continuous symmetry and conservation laws is usually referred to as Noether's theorem (Li and Xin 1991 and Li 1993-b). The study of symmetry of a system is usually based on the examination of the Lagrangian in configuration space, where the classical Noether theorem and their generalization where formulated in terms of Lagranges variables. A system with a singular Lagrangian that is subject to some phase space constraints is called a constrained Hamiltonian system.

The Lagrangian and Hamiltonian formulation for constrained systems is usually made through a formalism developed by Dirac (Dirac 1964). In this formalism Dirac showed that the algebra of Poisson brackets determines a division of constraints into two classes: so-called first-class and second-class constraints. The first-class constraints are those that have zero Poisson brackets with all other constraints in the subspace of phase space in which constraints hold; constraints which are not first-class are by definition second-class. The presence of constraints in such theories requires care when applying Dirac's method, especially when first class-constraints arise, since the first class constraints are generators of gauge transformations that lead to the gauge freedom and one should impose gauge-fixing condition for each first class constraints, which are not always an easy task. Besides, there have been objections to Dirac's conjecture, that is to obtain the equations of motion from the extended Hamiltonian H_E (Cabo 1986, Li 1991, Li 1993-a, Li 1993-c, Gogilidze et al 1994, Li 1994, Jin and Li 2001 and Eqab and Güler 1995). Several examples (Cawley 1979, Cawley 1980 and Frenkel 1980) were provided, where constraints are re-written in linearized form, including that Dirac's conjecture is invalid.

Recently a new method based on the Hamilton-Jacobi approach (Muslih and Güler 1995, 1998, Güler 1992-a, 1992-b, 1987, 1989 and Muslih 2002, 2003-a, 2003-b) has been developed to investigate constrained systems. In this method the equations of motion are obtained as total differential equations in many variables which require the investigation of integrability conditions, in other words, the integrability conditions may lead to new conditions. The equivalent Lagrangian method (Güler 1987 and Güler 1989) is used to obtain the set of Hamilton-Jacobi partial Differential Equation (HJPDE). Also in this approach the distinction between first and second-class constraints is not necessary.

The aim of this paper is to analyze constrained systems with singular higher order Lagrangians using the Hamilton-Jacobi method and the Dirac's method, and to show that the canonical equations of motion are obtained from the canonical set of Hamilton-Jacobi partial differential equations.

AN INVESTIGATION OF CONSTRAINED SYSTEMS

In this section, we shall investigate constrained Hamiltonian systems by using two methods: Dirac's method and the Hamilton Jacobi method.

Dirac's Conjecture For Systems With Singular Higher-Order Lagrangians

In this subsection we shall review the Dirac's conjecture for constrained systems (Jin and Li 2001 and Costa et al 1985). Let us consider a system with a singular higher-order Lagrangian given by

$$\mathrm{L} = \mathrm{L}(t, q_{(0)}, q_{(1)}, q_{(2)}, ..., q_{(n)}), \quad q = [q^1, q^2, ..., q^n], q_0 = q,$$

$$q_s = \frac{d^s}{dt^s} q(t) \tag{1}$$

In Ostrogradskii's formula, the momenta conjugated respectively to $q^i_{(n)}$

and $q^i_{(s)}$ are defined as

$$p_i^{(n-1)} = \frac{\partial L}{\partial q^i_{(n)}}, \quad (i = 1, 2, ..., n), \tag{2}$$

$$p_i^{(s-1)} = \frac{\partial L}{\partial q^i_{(s)}} - \dot{p}_i^{(s)}, \quad (s = 1, 2, ..., n-1), \tag{3}$$

using these relations one can go over from the Lagrangian description to the Hamiltonian description. The canonical Hamiltonian H_c is defined as

$$H_c = \sum_{s=0}^{(N-1)} p_i^{(s)} q^i_{(S+1)} - L(t, q_{(0)}, q_{(1)}, q_{(2)}, ..., q_{(n)}), \tag{4}$$

(Einstein's summation rule is used through out this paper). The extended Hess matrix reads

$$A^{ij}_{(n)} = \frac{\partial^2 L}{\partial q^i_{(n)} \partial q^j_{(n)}}, \quad i, j = 1, 2, ..., n. \tag{5}$$

For a regular system, the Hessian has rank n and the canonical coordinates are independent. For singular Lagrangian case the Hessian has rank $n-r$, $r < n$, and one cannot solve all $q^i_{(n)}$ from equation (2). In this case r of the momenta are dependent and we have the primary constraints

$$H'^0_\alpha(t, q^i_{(s)}, p_i^{(s)}) \approx 0, \quad (\alpha = 1, 2, ..., r), \tag{6}$$

in the phase space, where ≈ (weakly equality) means equality on the constrained hyper-surface.

Following reference (Jin and Li 2001), we obtain for any infinitesimal transformations, the change for any mechanical quantity F(t, q, p) as

$$\delta F = \varepsilon^\alpha \{F, \Phi_\alpha\}, \tag{7}$$

where ε^α are transformation parameters and Φ_α are the linear combination of the first-class constraints which may contain the primary first-class constraints (PFCC)- and the secondary first-class constraints (SFCC). The commutator { , } denotes for Poisson's brackets.

Dirac's conjecture for systems with a singular higher-order Lagrangian, states that all the first-class constraints are independent generators of the gauge transformations, consequently, the Hamiltonian dynamics has gauge invariance. If this conjecture holds true, then the dynamics of a system possessing primary $H'^0_\alpha \approx 0$, ($\alpha = 1, 2, ..., r$) and secondary $\phi^{(k)}_\alpha \approx 0$, ($\alpha = 1, 2, ..., r$, $k = 1, 2, ... m$) first class constraints should be correctly described by the equations of motion derived from the extended Hamiltonian (Jin and Li 2001 and Costa et al 1985).

$$H_E = H_c + \lambda^\alpha H'^0_\alpha + \lambda^\alpha_k \phi^{(k)}_\alpha, \tag{8}$$

where H_c is the canonical Hamiltonian , λ^α and λ^α_k are Lagrange multiplies.

The Hamilton-Jacobi Method

In papers (Muslih and Güler 1995, 1998, Güler 1992-a, 1992-b, 1987, 1989 and Muslih 2002, 2003-a, 2003-b) the Hamilton-Jacobi formulation of constrained systems has been studied. The starting point of this method is to consider the Lagrangian L = L (q_i, q_i, t), with the Hess matrix A_{ij}, (i, j = 1, ..., n), of rank (n–r), r < n.

The set of Hamilton-Jacobi partial differential equations [HJPDE] is expressed as (Güler 1992-a and Güler 1992-b)

$$H'_\alpha\left(x_\beta, q_a, \frac{\partial S}{\partial q_a}, \frac{\partial S}{\partial x_a}\right) = 0, \quad \alpha, \beta = 0, n-r+1, ..., n; \quad a = 1, ..., n-r, \tag{9}$$

where

$$H'_\alpha = H_\alpha (t_\beta, q_a, p_a) + p_\alpha, \tag{10}$$

and S being the action. The canonical Hamiltonian H_0 is defined as

$$H_0 = p_a\, w_a + p_\mu \,|_{p\nu = -H\nu} - L(t, q_i), \quad \mu, \nu = n-r+1, ..., n. \tag{11}$$

The equations of motion are obtained as total differential equations in many variables as follows (Güler 1992-a and Güler 1992-b):

$$dq_a = \frac{\partial H'_\alpha}{\partial p_a} dx_\alpha, \quad dp_a = \frac{\partial H'_\alpha}{\partial q_a} dx_\alpha, \qquad dp_\beta = \frac{\partial H'_\alpha}{\partial x_\beta} dx_\alpha, \tag{12}$$

$$dz = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}\right) dx_\alpha, \tag{13}$$

where z = S(x_α; q_a). These equations are integrable if and only if (Muslih and Güler 1995, 1998, Muslih 2002, 2003-a and 2003-b)

$$dH'_0 = 0, \tag{14}$$

$$dH'_\mu = 0, \quad \mu = n-r+1, ..., n. \tag{15}$$

If conditions (14), (15) are not satisfied identically, one considers them as new constraints and again testes the consistency conditions. Hence, the equations of motion reveal the fact that the Hamiltonians H'_α are considered as the infinitesimal generators of canonical transformations given by parameters x_α and the set of canonical phase-space coordinates q_a and p_a is obtained as functions of x_α. In this case for any mechanical quantity F(t, q, p), we have (Muslih and Güler 1995 and 1998)

$$\delta F = \varepsilon^\alpha [F, H'_\alpha], \quad \alpha = 0, n-r+1, ..., n, \tag{16}$$

where ε^α are arbitrary parameters.

One should notice that the equations of motion are derived from the multi-Hamiltonians H'_α in the Hamilton-Jacobi formalism and not from the extended Hamiltonian H_E as in the Dirac's formalism.

EXAMPLES

In this section, we shall solve four singular systems by the Dirac's method (Dirac 1964) and then by the Hamilton-Jacobi method (Muslih and Güler 1995, 1998, Güler 1992-a, 1992-b, 1987, 1989 and Muslih 2002, 2003-a, 2003-b).

Let us consider the singular Lagrangian (Jin and Li 2001 and Li 1993-a)

$$L = \sum_{s=1}^{N} \left(x_{(s)} z_{(s)} - y_{(s-1)} z_{(s)} \right) + xz. \tag{17}$$

The Lagrangian (17) is invariant under scale transformation

$$x'_{(s)} = \rho^{-1} x_{(s)}, \qquad y'_{(s)} = \rho^{-1} y_{(s)}, \qquad z'_{(s)} = \rho\, z_{(s)},$$
$$p'^{(N-1)}_x = \rho\, p^{(N-1)}_x, \qquad p'^{(N-1)}_z = \rho^{-1} p^{(N-1)}_z, \tag{18}$$

where ρ is a numerical parameter. According to the generalized canonical first Noether's theorem, one can obtain the conservation law (Jin and Li 2001 and Li 1993-a) as

$$p^{(s)}_x x_{(s)} + p^{(s)}_y y_{(s)} - p^{(s)}_z z_{(s)} = constant. \tag{19}$$

In a previous work (Muslih and El-zalan 2003-a) we consider the case for N = 3. In this work we considered the cases for N = 1, N = 2, N = 4 and the generalized case for any integer N.

Dirac's Approach

In this subsection, we shall study the conservation laws derived from H_E via the canonical Noether's first theorem to discuss the validity of Dirac's conjecture (Jin and Li 2001).

For N = 1, the singular Lagrangian for model (17) reads as

$$L = x_{(1)} z_{(1)} - y z_{(1)} + xz, \tag{20}$$

The generalized canonical momenta are given by

$$p^{(0)}_x = z_{(1)}, \qquad p^{(0)}_y = 0, \qquad p^{(0)}_z = x_{(1)} - y. \tag{21}$$

Since the rank of the Hessian is two, we have one primary constraint which can be expressed as

$$\phi^0 = p^{(0)}_y \approx 0. \tag{22}$$

The canonical Hamiltonian H_0 is defined as

$$H_0 = p^{(0)}_x \left(p^{(0)}_z + y \right) - xz. \tag{23}$$

Following Dirac one can define the total Hamiltonian H_T as

$$H_T = p^{(0)}_x \left(p^{(0)}_z + y \right) - xz + \lambda_1 p^{(0)}_y, \tag{24}$$

where λ_1 is a Lagrange multiplier. The consistency condition of the constraint (22) yields the following secondary constraints:

$$\phi^1 = \{\phi^0, H_T\} = -p_x^{(0)} \approx 0,$$
$$\phi^2 = \{\phi^1, H_T\} = z_{(0)} \approx 0,$$
$$\phi^3 = \{\phi^2, H_T\} = p_x^{(0)} \approx 0.$$

All the constraints are first class. If Dirac's conjecture holds true, the dynamics of this system should be described by the equations of motion arising from the extended Hamiltonian (8).

According to the generalized canonical Noether's first theorem (Li 1991 and Li 1993-b), the existence of the conservation law

$$p_x^{(0)} x_{(0)} + p_y^{(0)} y_{(0)} - p_z^{(0)} z_{(0)} = constant, \tag{25}$$

requires that all *SFCC*, $\phi^k \approx 0$, (k = 1, 2, 3), be invariant under the transformations (18). But it is obvious that the constraints cannot satisfy these conditions under transformations (18). Thus, the conservation law (25) could not be obtained from the extended Hamiltonian H_E. This implies that Dirac's conjecture is invalid for this model.

Now, the situation for N = 2, we get the singular Lagrangian (17) as

$$L = \sum_{s=1}^{2}\left(x_{(s)} z_{(s)} - y_{(s-1)} z_{(s)}\right) + xz. \tag{26}$$

The generalized canonical momenta are given by

$$p_x^{(0)} = z_{(1)} - z_{(3)}, \; p_y^{(0)} = -z_{(2)}, \; p_z^{(0)} = x_{(1)} - y - x_{(3)} + y_{(2)}, \tag{27}$$

$$p_x^{(1)} = z_{(2)}, \qquad p_y^{(1)} = 0, \qquad p_z^{(1)} = x_{(2)} - y_{(1)}. \tag{28}$$

The canonical Hamiltonian H_0 is defined as

$$H_0 = p_x^{(0)} x_{(1)} + p_y^{(0)} y_{(1)} + z_{(1)}\left(p_z^{(0)} - x_{(1)} + y\right) + p_x^{(1)}\left(p_z^{(1)} - y_{(1)}\right) - xz. \tag{29}$$

The primary constraint is

$$\phi^0 = p_y^{(1)} \approx 0. \tag{30}$$

Following Dirac, one can define the total Hamiltonian as:

$$H_T = H_0 + \lambda_2\, p_y^{(1)} \tag{31}$$

where λ_2 is a Lagrange multiplier. The consistency condition of the constraint (30) yields the following secondary constraints:

$$\phi^1 = \{\phi^0, H_T\} = -p_y^{(0)} - p_x^{(1)} \approx 0, \tag{32}$$

$$\phi^2 = \{\phi^1, H_T\} = -p_x^{(0)} \approx 0, \tag{33}$$

$$\phi^3 = \{\phi^2, H_T\} = -z_{(0)} \approx 0, \tag{34}$$

$$\phi^4 = \{\phi^3, H_T\} = -z_{(1)} \approx 0, \tag{35}$$

$$\phi^5 = \{\phi^4, H_T\} = -p_x^{(1)} \approx 0, \tag{36}$$

$$\phi^6 = \{\phi^5, H_T\} = p_x^{(0)} - z_{(1)} \approx 0, \tag{37}$$

which are all first class.

Now, the situation for $N = 4$, we get the singular Lagrangian (17) as

$$\mathrm{L} = \sum_{s=1}^{4}\left(x_{(s)} z_{(s)} - y_{(s-1)} z_{(s)}\right) + xz. \qquad (38)$$

The generalized canonical momenta are given by

$$p_x^{(0)} = z_{(1)} - z_{(3)}, \quad p_y^{(0)} = - z_{(2)} + z_{(4)}, \quad p_z^{(0)} = x_{(1)} - y - x_{(3)} + y_{(2)}, \qquad (39)$$

$$p_x^{(1)} = z_{(2)} - z_{(4)}, \quad p_y^{(1)} = z_{(5)} - z_{(3)}, \quad p_z^{(1)} = x_{(2)} - y_{(1)} - x_{(4)} + y_{(3)}. \qquad (40)$$

$$p_x^{(2)} = z_{(3)} - z_{(5)}, \quad p_y^{(2)} = - z_{(4)}, \quad p_z^{(2)} = x_{(3)} - y_{(2)} - x_{(5)} + y_{(4)}. \qquad (41)$$

$$p_x^{(3)} = z_{(4)}, \quad p_y^{(3)} = 0, \quad p_z^{(3)} = x_{(4)} - y_{(3)}. \qquad (42)$$

Since the rank of the Hessian is two, we have one primary constraint, which can be expressed as

$$\phi^0 = p_y^{(3)} \approx 0. \qquad (43)$$

The canonical Hamiltonian H_0 is defined as

$$H_0 = \sum_{s=1}^{3}\left[p_x^{(s-1)} x_{(s)} + p_y^{(s-1)} y_{(s)} + z_{(s)}\left(p_z^{(s-1)} - x_{(s)} + y_{(s-1)}\right)\right]$$
$$+ \; p_x^{(3)}\left(p_z^{(3)} + y_{(3)}\right) - xz. \qquad (44)$$

Following Dirac one can define the total Hamiltonian H_T as

$$H_T = H_0 + \lambda_4\, p_y^{(3)}, \qquad (45)$$

where λ_4 is a Lagrange multiplier. The condition of conservation of the constraint (43) in time yields the following secondary constraints (Li 1991, 1993-b and Jin and Li 2001):

$$\phi^1 = \{\phi^0, H_T\} = - p_x^{(3)} - p_y^{(2)} \approx 0, \qquad (46)$$

$$\phi^2 = \{\phi^1, H_T\} = p_x^{(2)} + p_y^{(1)} \approx 0, \qquad (47)$$

$$\phi^3 = \{\phi^2, H_T\} = - p_x^{(1)} - p_y^{(0)} \approx 0, \qquad (48)$$

$$\phi^4 = \{\phi^3, H_T\} = - p_x^{(0)} \approx 0, \qquad (49)$$

$$\phi^5 = \{\phi^4, H_T\} = z_{(0)} \approx 0, \qquad (50)$$

$$\phi^6 = \{\phi^5, H_T\} = z_{(1)} \approx 0, \qquad (51)$$

$$\phi^7 = \{\phi^6, H_T\} = z_{(2)} \approx 0, \qquad (52)$$

$$\phi^8 = \{\phi^7, H_T\} = z_{(3)} \approx 0, \qquad (53)$$

$$\phi^9 = \{\phi^8, H_T\} = p_x^{(3)} \approx 0, \qquad (54)$$

$$\phi^{10} = \{\phi^9, H_T\} = - p_x^{(2)} + z_{(3)} \approx 0, \qquad (55)$$

$$\phi^{11} = \{\phi^{10}, H_T\} = p_x^{(1)} - z_{(2)} + p_x^{(3)} \approx 0, \qquad (56)$$

$$\phi^{12} = \{\phi^{11}, H_T\} = - p_x^{(0)} + z_{(1)} - p_x^{(2)} \approx 0, \qquad (57)$$

$$\phi^{13} = \{\phi^{12}, H_T\} = p_x^{(1)} - z_{(0)} \approx 0, \qquad (58)$$

As in the first two examples all the constraints are first class constraints. One can simply show that the Dirac's conjecture is invalid for this modal.

Now, let us consider the general case for any positive integer N. The generalized canonical momenta with the condition $q^i_{(m)} = 0$ for all $m \geq N$, are given by

$$p_x^{(s-1)} = z_{(s)} - z_{(s+2)}, \tag{59}$$

$$p_y^{(s-1)} = -z_{(s+1)} + z_{(s+3)}, \tag{60}$$

$$p_z^{(s-1)} = x_{(s)} - y_{(s-1)} - x_{(s+2)} + y_{(s+1)}, \tag{61}$$

Now, the canonical Hamiltonian H_0 with the condition $q^i_{(m)} = 0$ for all $m \geq N$, is given by

$$H_0 = p_x^{(N-1)}\left[p_z^{(N-1)} + y_{(N-1)}\right] + \sum_{s=0}^{N-1}\left[p_i^{(s)} q^i_{(s+1)} + z_{(s+1)}\left(y_{(s)} - x_{(s+1)}\right)\right] - xy. \tag{62}$$

For any positive integer N, the primary constraint is

$$\phi_0 = p_y^{(N-1)} \approx 0. \tag{63}$$

Following Dirac one can define the total Hamiltonian H_T as

$$\mathrm{H_T} = \mathrm{H_0} + \lambda_\mathrm{N}\, p_y^{(N-1)} \tag{64}$$

where λ_N are Lagrange multipliers. The condition of conservation of the constraint (63) in time yields the secondary constraints given in (Li 1991, 1993-b and Jin and Li 2001).

All the constraints are first class constraints. If Dirac's conjecture holds true, the dynamics of the system should be described from the extended Hamiltonian H_E. According to the generalized canonical Noether's first theorem (Li 1991 and Li 1993-b), the existence of the conservation law

$$p_x^{(s)} x_{(s)} + p_y^{(s)} y_{(s)} - p_z^{(s)} z_{(s)} = \text{const}, \tag{65}$$

requires that all SFCC, $\phi^k \approx 0$ (k = 1, 2, ..., 3N+1), be invariant under the transformations (18). But it is obvious that the constraints cannot satisfy these conditions under transformations (18). Thus, the conservation law (65) could not be obtained from the extended Hamiltonian H_E. This implies that Dirac's conjecture is invalid for this model.

Hamilton-Jacobi Approach

Now we would like to investigate the same systems by the Hamilton-Jacobi method (Muslih and Güler 1995, 1998, Güler 1992-a, 1992-b, 1987, 1989 and Muslih 2002, 2003-a, 2003-b). For N = 1, equations (9), (21) and (23) leads us the following set of HJPDE:

$$H_0' = P_0 + p_x^{(0)}\left(p_z^{(0)} + y\right) - xz, \quad P_0 = \frac{\partial S}{\partial t}, \tag{66}$$

$$H_y'^{(0)} = p_y^{(0)} = 0, \quad p_y^{(0)} = \frac{\partial S}{\partial y}, \tag{67}$$

The equations of motion are obtained as total differential equations as follows:

$$dx = \left(p_z^{(0)} + y\right)dt, \; dz = p_x^{(0)} dt, \tag{68}$$

$$dp_x^{(0)} = zdt, \; dp_y^{(0)} = - p_x^{(0)} dt, \quad dp_z^{(0)} = xdt. \tag{69}$$

To check whether this set is integrable or not, let us consider the variations of constraints (66) and (67). In fact

$$dH_0' = \mathrm{F}_1 dy = p_x^{(0)} dy, \tag{70}$$

$$dH_2' = dp_y^{(0)} = -\mathrm{F}_1 dt. \tag{71}$$

Since F_1 is not identically zero, we consider it as a new constraint and will lead us to the following constraints:

$$d\mathrm{F}_1 = \mathrm{F}_2 dt = zdt, \tag{72}$$

and its variation is

$$d\mathrm{F}_2 = \mathrm{F}_3 dt = p_x^{(0)} dt \equiv 0. \tag{73}$$

The set of equations (68) and (69) is integrable and the canonical phase space coordinates are obtained in terms of parameters (t, y).

For N = 2, we obtain the set of HJPDE as follows:

$$H_0' = P_0 + p_x^{(0)} x_{(1)} + dp_y^{(0)} y_{(1)} + z_{(1)}\left(p_z^{(0)} - x_{(1)} + y\right)$$
$$+ p_x^{(1)} \left(p_z^{(1)} + y_{(1)}\right) - xz, \quad P_0 = \frac{\partial S}{\partial t}, \tag{74}$$

$$H_y'^{(1)} = p_y^{(1)} = 0, \quad p_y^{(1)} = \frac{\partial S}{\partial y_{(1)}}, \tag{75}$$

The equations of motion are obtained as total differential equations as follows:

$$dx = x_{(1)}dt, dy = y_{(1)}dt, \; dz = z_{(1)}dt, \tag{76}$$

$$dx_{(1)} = \left(p_z^{(1)} + y_{(1)}\right)dt, \; dz_{(1)} = p_x^{(1)} dt, \tag{77}$$

$$dp_x^{(0)} = zdt, \quad dp_y^{(0)} = - z_{(1)}dt, \; dp_z^{(0)} = xdt. \tag{78}$$

$$dp_x^{(1)} = \left(z_{(1)} - p_x^{(0)}\right)dt, \quad dp_y^{(1)} = \left(p_y^{(0)} + p_x^{(1)}\right)dt, \quad dp_z^{(1)} = \left(x_{(1)} - p_z^{(0)} - y\right)dt. \tag{79}$$

The total variations of the constraints (74) and (75) lead us to obtain

$$dH_0' = F_1 dy_{(1)} = \left(p_y^{(0)} + p_x^{(1)}\right)dy_{(1)}, \tag{80}$$

$$dH_2' = dp_y^{(1)} = -F_1 dt. \tag{81}$$

Since F_1 is not identically zero, we consider it as a new constraint and it is variation leads to following constraints

$$dF_1 = F_2 dt = -\, p_x^{(0)}\, dt = 0, \quad (82)$$

$$dF_2 = F_3 dt = -\, z_{(0)} dt = 0, \quad (83)$$

$$dF_3 = F_4 dt = -\, z_{(1)} dt = 0, \quad (84)$$

$$dF_4 = F_5 dt = -\, p_x^{(1)}\, dt = 0, \quad (85)$$

$$dF_5 = F_6 dt = \left(p_x^{(0)} - z_{(1)}\right) dt = 0. \quad (86)$$

The total variation of F_6 is identically zero, hence the canonical phase space coordinates are obtained in terms of parameters $(t, y_{(1)})$.

Now, for $N = 4$, we obtain the set of HJPDE as follows:

$$H_0' = P_0 + \sum_{s=1}^{3} \left[p_x^{(s-1)} x_{(s)} + p_y^{(s-1)} y_{(s)} + z_{(s)} \left(p_z^{(s-1)} - x_{(s)} + y_{(s-1)} \right) \right]$$

$$+\; p_x^{(3)} \left(p_z^{(3)} + y_{(3)} \right) - xz, \quad P_0 = \frac{\partial S}{\partial t} \quad (87)$$

$$H_y'^{(3)} = p_y^{(3)} = 0, \quad p_y^{(3)} = \frac{\partial S}{\partial y_{(3)}}, \quad (88)$$

The equations of motion are obtained as follows:

$$dx = x_{(1)} dt, dy = y_{(1)} dt,\ dz = z_{(1)} dt, \quad (89)$$

$$dx_{(1)} = x_{(2)} dt,\ dy_{(1)} = y_{(2)} dt,\ dz_{(1)} = z_{(2)} dt, \quad (90)$$

$$dx_{(2)} = x_{(3)} dt,\ dy_{(2)} = y_{(3)} dt,\ dz_{(2)} = z_{(3)} dt, \quad (91)$$

$$dx_{(3)} = \left(p_z^{(3)} + y_{(3)}\right) dt,\ dz_{(3)} = p_x^{(3)}\, dt, \quad (92)$$

$$dp_x^{(0)} = z dt,\ dp_y^{(0)} = -\, z_{(1)} dt,\ dp_z^{(0)} = x dt. \quad (93)$$

$$dp_x^{(1)} = \left(z_{(1)} - p_x^{(0)}\right) dt,\ \ dp_y^{(1)} = -\left(p_y^{(0)} + z_{(2)}\right) dt,\ dp_z^{(1)} = \left(x_{(1)} - p_z^{(0)} - y\right) dt. \quad (94)$$

$$dp_x^{(2)} = \left(z_{(2)} - p_x^{(1)}\right) dt, dp_y^{(2)} = -\left(p_y^{(1)} + z_{(3)}\right) dt, dp_z^{(2)} = \left(x_{(2)} - p_z^{(1)} - y_{(1)}\right) dt. \quad (95)$$

$$dp_x^{(3)} = \left(z_{(3)} - p_x^{(2)}\right) dt,\ dp_y^{(3)} = -\left(p_y^{(2)} + p_x^{(3)}\right) dt,\ dp_z^{(3)} = \left(x_{(3)} - p_z^{(2)} - y_{(2)}\right) dt. \quad (96)$$

To check whether the set of equations (89)-(96) is integrable or not, let us consider the variation of equations (87) and (88), we get

$$dH_0' = F_1 dy_{(3)} = \left(p_y^{(2)} + p_x^{(3)}\right) dy_{(3)}, \quad (97)$$

$$dH_2' = dp_y^{(2)} = -F_1 dt. \quad (98)$$

Since F_1 is not identically zero, we consider it as a new constraint and it is total variation leads to following constraints

$$dF_1 = F_2 dt = -\left(p_y^{(1)} + p_x^{(2)}\right) dt = 0, \quad (99)$$

$$dF_2 = F_3 dt = -\left(p_y^{(0)} + p_x^{(1)}\right) dt = 0, \quad (100)$$

$$dF_3 = F_4 dt = -\, p_x^{(0)}\, dt = 0, \quad (101)$$

$$dF_4 = F_5 dt = z dt = 0, \quad (102)$$

$$dF_5 = F_6 dt = z_{(1)} dt = 0, \quad (103)$$

$$dF_6 = F_7 dt = z_{(2)} dt = 0, \quad (104)$$

$$dF_7 = F_8 dt = z_{(3)} dt = 0, \quad (105)$$

$$dF_8 = F_9 dt = p_x^{(3)} dt = 0, \quad (106)$$

$$dF_9 = F_{10} dt = \left(z_{(3)} - p_x^{(2)}\right) dt = 0, \quad (107)$$

$$dF_{10} = F_{11} dt = \left(z_{(2)} - p_x^{(1)}\right) dt = 0, \quad (108)$$

$$dF_{11} = F_{12} dt = \left(z_{(1)} - p_x^{(0)}\right) dt = 0, \quad (109)$$

The total variation of F_{12} is identically satisfied, hence the system is integrable and the canonical phase space coordinates are obtained in terms of parameters $(t, y_{(3)})$.

Finally, let us consider the general case for any positive integer N. Making use of equations (9), (62) and (63), we obtain the set of HJPDE as follows:

$$H_0' = P_0 + p_x^{(N-1)} \left[p_z^{(N-1)} + y_{(N-1)}\right]$$

$$+ \sum_{s=0}^{N-1} \left[p_i^{(s)} q_{(s+1)}^i + z_{(s+1)} \left(y_{(s)} - x_{(s+1)}\right)\right] - xy, \qquad P_0 = \frac{\partial S}{\partial t} \quad (110)$$

$$H_y'^{(N-1)} = p_y^{(N-1)} = 0, \quad p_y^{(N-1)} = \frac{\partial S}{\partial y_{(N-1)}}, \quad (111)$$

The equation of motion are obtained as total differential as follows:

$$dq_{(l-1)}^i = q_{(l)}^i dt, \; dp_y^{(l-1)} = -\left(z_{(l)} + p_y^{(l-2)}\right) dt, \; l = 1, \ldots, N-1, \quad (112)$$

$$dx_{(N-1)} = \left(p_z^{(N-1)} + y_{(N-1)}\right) dt, dz_{(N-1)} = p_x^{(N-1)} dt, \; dp_y^{(N-1)} = -\left(p_x^{(N-1)} + p_y^{(N-2)}\right) dt \quad (113)$$

$$dp_x^{(k-1)} = \left(z_{(k-1)} - p_x^{(k-2)}\right) dt, \; dp_z^{(k-1)} = \left(x_{(k-1)} - p_z^{(k-2)} - y_{(k-2)}\right) dt, \; k=1,.., N. \quad (114)$$

To check whether the set of equations (112)-(114) is integrable or not, let us consider the variation of equations (110) and (111). In fact

$$dH_0' = F_1 dy_{(N-1)} = \left(p_x^{(N-1)} + p_y^{(N-2)}\right) dy_{(N-1)}, \quad (115)$$

$$dH_2' = dp_y^{(N-1)} = -F_1 dt. \quad (116)$$

Since F_1 is not identically zero, we consider it as a new constraint and it is total variation leads to following $3N$ constraints

$$dF_1 = F_2 dt = -\left(p_x^{(N-2)} + p_y^{(N-3)}\right) dt = 0, \quad (117)$$

$$dF_2 = F_3 dt = -\left(p_x^{(N-3)} + p_y^{(N-4)}\right) dt = 0, \quad (118)$$

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$$dF_{N-1} = F_N dt = -p_x^{(0)} dt = 0, \quad (119)$$

$$dF_N = F_{N+1} dt = z dt = 0, \quad (120)$$

$$dF_{N+1} = F_{N+2} dt = -z_{(1)} dt = 0, \quad (121)$$

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$$dF_{2N-1} = F_{2N}dt = z_{(N-1)}dt = 0, \tag{122}$$

$$dF_N = F_{2N+1}dt = p_x^{(N-1)}dt = 0, \tag{123}$$

$$dF_{2N+1} = F_{2N+2}dt = \left(z_{(N-1)} - p_x^{(N-2)}\right)dt = 0, \tag{124}$$

$$dF_{3N+1} = F_{2N+3}dt = \left(z_{(N-2)} - p_x^{(N-3)}\right)dt = 0, \tag{125}$$

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$$dF_{3N-1} = F_{3N}dt = \left(z_{(1)} - p_x^{(0)}\right)dt = 0, \tag{126}$$

One should notice that the total variation of F_{3N} is

$$DF_{3N} = F_N dt, \tag{127}$$

which is identically satisfied. Hence, the system is integrable and the canonical phase space coordinates are obtained in terms of parameters (t, $y_{(N-1)}$).

CONCLUSION

In this work we have investigated constrained systems using the Hamilton-Jacobi method, where this method is completely different from the Dirac's method treatment of constrained system. In the Hamilton-Jacobi-method, the equations of motion are obtained from the multi-Hamiltonians H'_α which are considered as infinitesimal generators of canonical transformations given by parameters x_α and these Hamiltonians are invariant under the transformations (18). While in the Dirac's method the equations of motion are obtained from the extended Hamiltonian H_E, which includes all the primary first class constraints (PFCC) and secondary first class constraints (SFCC), where some of them in the given examples are not invariant under the transformations (18) (Li 1991, Li 1993-b and Jin and Li 2001). In this case one cannot deduce all the generalized canonical equations arising from the extended Hamiltonian H_E with the generalized Poincare- Cartan integral invariant (Domici and Gomis 1980 and Guo et al 2001). In this case the Dirac's conjecture is invalid. On other hand, the comparison between the equations of motion obtained by using Dirac's method and the Hamilton-Jacobi method have been discussed in previous work (Muslih and El-Zalan 2003-b) and a complete agreement has been observed.

As a conclusion, we notice that the Dirac's conjecture is invalid for systems with higher order Lagrangian (17), but the Hamilton-Jacobi method gives the correct integrable generalized equations of motion for the given systems.

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